FOURIER ANALYSIS. (fall 2016)

MODEL SOLUTIONS FOR SET 3

Exercise 1. **(i)** Let $N \in \mathbb{N}$. Show that there exists a non-trivial function $f \in L^1[-\pi, \pi]$ such that $F_N * f(x) = 0$ for all x.

(ii) Is there a non-trivial function $f \in L^1[-\pi, \pi]$ so that $F_N * f(x) = 0$ for all x and for *all* $N \geq 0$?

Solution 1. **(i)** We can choose the function f as $f(x) = e^{i(N+1)x}$. Now we have $\widehat{f}(n) = 0$ for any $n \neq N + 1$ and so

$$F_N * f(x) = \sum_{n=-(N-1)}^{N-1} \Big(\frac{N - |n|}{N}\Big) \widehat{f}(n) e^{inx} = 0.$$

(ii) There is no such function. As the Fejer kernels are a good sequence of kernels, we know that $\|f - F_N * f\|_{L^1(-\pi,\pi)}$ goes to zero as N goes to infinity. Assuming that $F_N * f(x) = 0$ for all x and for all N gives us

$$\|f\|_{L^1(-\pi,\pi)} = \|f - F_N * f\|_{L^1(-\pi,\pi)} \to 0.$$

This means that $\|f\|_{L^1(-\pi,\pi)} = 0$, so $f(x) = 0$ for almost every x.

Exercise 2. Use the results of lectures and verify that the Fourier-series of an integrable function converges at any point of differentiability of f.

Solution 2. Let $f \in L^1(-\pi, \pi)$ be 2π-periodic and differentiable at x_0. We apply Dini's criterion. We need to show that

$$\int_0^\pi \left| \frac{f(x_0 + t) + f(x_0 - t)}{2} - f(x_0) \right| \frac{dt}{t} < \infty.$$

We can write this equivalently as

$$\int_0^\pi \frac{1}{2} \left| \frac{f(x_0 + t) - f(x_0)}{t} + \frac{f(x_0 - t) - f(x_0)}{t} \right| dt < \infty.$$

As f is differentiable at x_0 there exists $\delta > 0$ such that for $0 < t < \delta$ we have

$$\max \left\{ \left| \frac{f(x_0 + t) - f(x_0)}{t} \right|, \left| \frac{f(x_0 - t) - f(x_0)}{t} \right| \right\} < |f'(x_0)| + 1.$$

We can then estimate

$$\begin{aligned} &\int_0^\pi \frac{1}{2} \left| \frac{f(x_0 + t) - f(x_0)}{t} + \frac{f(x_0 - t) - f(x_0)}{t} \right| dt \\ &\leq \int_0^\delta \frac{1}{2}(2|f'(x_0)| + 2)\, dt + \int_\delta^\pi \frac{1}{2\delta}(|f(x_0 + t)| + 2|f(x_0)| + |f(x_0 - t)|)\, dt \\ &\leq \delta(|f'(x_0)| + 1) + \frac{2\|f\|_{L^1}}{\delta} < \infty. \end{aligned}$$

Exercise 3. Suppose the sequence $(x_n)_{n=1}^{\infty}$ is equidistributed (mod 1) and $a \in \mathbb{Z} \setminus \{0\}$. Show that then also the sequence $(ax_n)_{n=1}^{\infty}$ is equidistributed (mod 1).

Does the result hold when $\alpha \notin \mathbb{Q}$?

Solution 3. We use Weyl's criterion. Suppose (x_n) is equidistributed mod 1. Then by the criterion the equality

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} e^{2\pi i l x_k} = 0$$

holds for every $l \in \mathbb{Z} \setminus \{0\}$. If it holds for all such l, it also holds for $l = al'$, where l' runs over the nonzero integers. Thus it holds also that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} e^{2\pi i a l' x_k} = 0 \qquad \text{for all } l' \in \mathbb{Z} \setminus \{0\}$$

which shows that (ax_n) is equidistributed by Weyl's criterion.

If a is allowed to be an irrational number, the result doesn't hold. For example, the sequence $(n\sqrt{2})$ is equidistributed mod 1 by Corollary 5.4 of the lecture notes, but the sequence $(\sqrt{(2)}n/\sqrt{2}) = (n) = (0)$ is not.

Extra: For $a \in \mathbb{Q} \setminus \mathbb{Z}$ the result is also not true, since if (x_n) is a sequence of numbers on the interval $[0, 1]$ that is equidistributed, then the sequence $(x_n/2)$ lies only on the interval $[0, 1/2]$ and is therefore not equidistributed.

Exercise 4. Show that the sequence $(< a \log n >)_{n=1}^{\infty}$ is *not* equidistributed (mod 1) for any $a \in \mathbb{R}$.

Solution 4. Let a be a real number. If a is zero then obviously $(a \log n)$ is not equidistributed, so assume $a \neq 0$. We then apply Weyl's criterion. It needs to be shown that for some non-zero integer k the sequence of partial sums

$$\frac{1}{N} \sum_{n=1}^{N} e^{2\pi i a k \log n}$$

does not converge to zero as $N \to \infty$. We will show this for $k = 1$. We now simplify that

$$\frac{1}{N} \sum_{n=1}^{N} e^{2\pi i a \log n} = \frac{1}{N} \sum_{n=1}^{N} n^{2\pi i a}$$

The corresponding integral is

$$\frac{1}{N} \int_1^{N+1} x^{2\pi i a} dx = \frac{1}{(2\pi i a + 1)N} \left((N+1)^{2\pi i a + 1} - 1 \right).$$

This diverges as $N \to \infty$, since the numbers $(N+1)^{2\pi ia}$ are on the unit circle and go around it infinitely often as $N \to \infty$. We now compare the sum to the integral. If we can prove that the difference

$$d_N = \frac{1}{N}\int_1^{N+1} x^{2\pi ia}dx - \frac{1}{N}\sum_{n=1}^{N} n^{2\pi ia}$$

converges to zero as $N \to \infty$, then it will follow that the original sum cannot converge to zero. We calculate that

$$\begin{aligned} d_N &= \frac{1}{N}\int_1^{N+1} x^{2\pi ia}dx - \frac{1}{N}\sum_{n=1}^{N} n^{2\pi ia} \\ &= \frac{1}{N}\sum_{n=1}^{N}\left(\int_n^{n+1} x^{2\pi ia}dx - n^{2\pi ia}\right) \\ &= \frac{1}{N}\sum_{n=1}^{N}\left(\frac{(n+1)^{2\pi ia+1} - n^{2\pi ia+1}}{2\pi ia+1} - n^{2\pi ia}\right) \\ &= \frac{1}{N}\sum_{n=1}^{N}\frac{1}{2\pi ia+1}\left(\frac{(1+1/n)^{2\pi ia+1}-1}{1/n} - 1\right)n^{2\pi ia} \end{aligned}$$

We now estimate the terms of this sum when n is large. If n is large, $1/n$ is small and we can use the Taylor series of the function

$$f(x) = \frac{1}{2\pi ia+1}(1+x)^{2\pi ia+1}$$

at the point $x = 0$ to find the estimate

$$\left|\frac{f(x)-f(0)}{x} - f'(0)\right| \le Cx.$$

This amounts to

$$\left|\frac{1}{2\pi ia+1}\frac{(1+1/n)^{2\pi ia+1}-1}{1/n} - 1\right| \le \frac{C}{n}.$$

We now use this together with the fact that $|n^{2\pi ia}| = 1$ when $a \in \mathbb{R}$ to estimate that

$$\left|\frac{1}{2\pi ia+1}\left(\frac{(1+1/n)^{2\pi ia+1}-1}{1/n} - 1\right)n^{2\pi ia}\right| \le \frac{C}{n}$$

for all positive integers n. Note that while we only proved this for large enough n, we can choose C big enough so that it holds for smaller values as well. We then get that

$$\frac{1}{N}\sum_{n=1}^{N}\left|\frac{1}{2\pi ia+1}\left(\frac{(1+1/n)^{2\pi ia+1}-1}{1/n} - 1\right)n^{2\pi ia}\right| \le \frac{1}{N}\sum_{n=1}^{N}\frac{C}{n} \le \frac{C_2\log N}{N}.$$

The right hand side converges to zero as $N \to \infty$. Hence the sum

$$\frac{1}{N} \sum_{n=1}^{N} \frac{1}{2\pi ia+1} \left(\frac{(1+1/n)^{2\pi ia+1}-1}{1/n} - 1 \right) n^{2\pi ia}$$

also converges to zero as $N \to \infty$.

Exercise 5. Prove Corollary 4.8; that is, show that if a 2π-periodic function $f(x)$ is piecewise C^1, then its Fourier series converges at every point, and

$$\lim_{N\to\infty} S_N f(x) = \lim_{t\to 0} \frac{f(x+t)+f(x-t)}{2}, \qquad x \in [-\pi, \pi].$$

Solution 5. Let f be piecewise C^1, so there exists a partition $-\pi = a_0 < a_1 < \ldots < a_n = \pi$ such that the restrictions $f|(a_i, a_{i+1})$ are C^1 and one-sided limits Using Exercise 2 we only need to show the convergence at First of all, we lose no generality if we only show the convergence at the point $x_0 = 0$, since we can always make a substitution $F(x) = f(x+x_0)$ as in the proof of the Dini condition. We hence want to show

$$\lim_{N\to\infty} S_N f(0) = \frac{f(0+)+f(0-)}{2}, \tag{1}$$

where $f(0+)$ and $f(0-)$ denote the left and right limits of the function f at the point $x = 0$. For a piecewise C^1-function f these limits always exist but may disagree. We now apply Lemma 4.2 of the lecture notes. By the lemma, to show that (1) holds it is enough to check that

$$\int_0^\pi \left| \frac{f(x)+f(-x)}{2} - \frac{f(0+)+f(0-)}{2} \right| \frac{dx}{x} < \infty.$$

It will be enough to show that the functions

$$g_+(x) = \frac{f(x)-f(0+)}{x} \quad \text{and} \quad g_-(x) = \frac{f(-x)-f(0-)}{x}$$

are bounded on the interval $(0, \pi)$, since

$$\int_0^\pi \left| \frac{f(x)+f(-x)}{2} - \frac{f(0+)+f(0-)}{2} \right| \frac{dx}{x} = \frac{1}{2} \int_0^\pi \left| \frac{f(x)-f(0+)}{x} + \frac{f(0+)+f(0-)}{2} \right| dx.$$

Let us take $\epsilon > 0$ sufficiently small so that f is C^1 on the interval $(0, \epsilon]$. Then for $\epsilon \le x \le \pi$ we have that

$$|g_+(x)| \le \frac{|f(x)-f(0+)|}{\epsilon},$$

which is bounded since f is bounded. When $0 < x < \epsilon$, we use the mean value theorem to find, for each x, a point $\xi_x \in (0, x)$ such that

$$g_+(x) = \frac{f(x)-f(0+)}{x} = f'(\xi_x)$$

Since f is piecewise C^1, by definition the derivative $f'(x)$ is continuous on the closed interval $[0, \epsilon]$ and hence bounded. This shows that $g_+(x)$ is bounded, and by the same arguments so is $g_-(x)$. Hence the result is proven.

Exercise 6. Let $f \in C_{\#}(-\pi, \pi)$. Assume that f has another period $\beta > 0$: $\quad f(\beta + x) = f(x)$ for all x. Show that if f is constant if $\beta/2\pi$ is irrational.

Solution 6. Define a new function g by setting $g(x) = f(x + \beta)$. By Exercise 4 from Set 1, we know that $\widehat{g}(n) = e^{in\beta}\widehat{f}(n)$. But as $g(x) = f(x+\beta) = f(x)$ for all x, then we also have $\widehat{g}(n) = \widehat{f}(n)$.

As $\beta/2\pi$ is irrational, we know that for any integers n and m we have $n\beta \neq 2m\pi$ unless $n = m = 0$. This means that $e^{in\beta} \neq 1$ for any $n \neq 0$ and we can solve

$$(1 - e^{in\beta})\widehat{f}(n) = 0 \text{ so } \widehat{f}(n) = 0.$$

As the function $h(x) = \widehat{f}(0)$ has the same Fourier coefficients as f, we know that $f(x) = \widehat{f}(0)$ almost everywhere. As f was continuous, we have shown that f is constant.

Extra: If we only assume $f \in L^1(-\pi, \pi)$, then the previous computation still shows that $f(x) = \widehat{f}(0)$ almost everywhere. However, if we define set $S = \{2n\pi + m\beta : n, m \in \mathbb{Z}\}$, then the characteristic function of S is both 2π-periodic and β-periodic, but is not a constant function.

Exercise 7*. Is the sequence $(\sqrt[3]{n})_{n=1}^{\infty}$ equidistributed mod 1?

Solution 7*. We will show that the sequence is equidistributed mod 1 directly from the definition. Let $(p, q) \subset [0, 1)$. Then $\langle \sqrt[3]{n} \rangle \in (p, q)$ when $\sqrt[3]{n} \in (p+k, q+k)$ for some $k \geq 0$. Equivalently

$$n \in (k^3 + 3pk^2 + 3p^2k + p^3, k^3 + 3qk^2 + 3q^2k + q^3).$$

We denote

$$A_N = \frac{1}{N}|\{\langle \sqrt[3]{n} \rangle \in (p, q) : 1 \leq n \leq N\}|.$$

Now A_N increases when N goes through numbers in interval $(k^3 + 3pk^2 + 3p^2k + p^3, k^3 + 3qk^2 + 3q^2k + q^3)$ for some k and decreases elsewhere. This means that A_N has a local maximum when $N = \lceil k^3 + 3qk^2 + 3q^2k + q^3 \rceil - 1$, just before it starts decreasing. When $N = \lceil k^3 + 3qk^2 + 3q^2k + q^3 \rceil - 1$ we have

$$\begin{aligned}
A_N &\leq \frac{\sum_{l=0}^{k}(l^3 + 3ql^2 + 3q^2l + q^3 - (l^3 + 3pl^2 + 3p^2l + p^3) + 1)}{N} \\
&= \frac{(q-p)k(k+1)(2k+1)/2 + (q^2 - p^2)3k(k+1)/2 + (k+1)(q^3 - p^3 + 1)}{N} \\
&\leq \frac{(q-p)k(k+1)(k+\frac{1}{2}) + (q^2 - p^2)3k(k+1)/2 + (k+1)(q^3 - p^3 + 1)}{k^3 + 3qk^2 + 3q^2k + q^3 - 2}
\end{aligned}$$

As this expression is a rational function with both denominator and numerator having degree 3, we see that

$$
\begin{aligned}
A_N &\le \frac{(q-p)k(k+1)(k+\frac{1}{2}) + (q^2-p^2)3k(k+1)/2 + (k+1)(q^3-p^3+1)}{k^3+3qk^2+3q^2k+q^3-2} \\
&= \frac{(q-p)(1+k^{-1})(1+\frac{1}{2}k^{-1}) + (q^2-p^2)3k^{-1}(1+k^{-1})/2 + (k^{-2}+k^{-3})(q^3-p^3+1)}{1+3qk^{-1}+3q^2k^{-2}+q^3k^{-3}-2k^{-3}} \\
&\to q-p
\end{aligned}
$$

This means that $\limsup_{N\to\infty} A_N \le q-p$. Similarly A_N has a local minimum whenever $N = \lfloor k^3+3pk^2+3p^2k+p^3 \rfloor$. Then

$$
\begin{aligned}
A_N &\ge \frac{\sum_{l=0}^{k-1}(l^3+3ql^2+3q^2l+q^3-(l^3+3pl^2+3p^2l+p^3)-1)}{N} \\
&= \frac{(q-p)k(k-1)(2k-1)/2 + (q^2-p^2)3k(k-1)/2 + k(q^3-p^3-1)}{N} \\
&\ge \frac{(q-p)k(k-1)(k-\frac{1}{2}) + (q^2-p^2)3k(k-1)/2 + k(q^3-p^3-1)}{k^3+3pk^2+3p^2k+p^3} \\
&= \frac{(q-p)(1-k^{-1})(1-\frac{1}{2}k^{-1}) + (q^2-p^2)3k^{-1}(1-k^{-1})/2 + k^{-2}(q^3-p^3+1)}{1+3pk^{-1}+3p^2k^{-2}+p^3k^{-3}} \\
&\to q-p
\end{aligned}
$$

This shows that $\liminf_{N\to\infty} A_N \ge q-p$. Combining the results, we see that $\lim_{N\to\infty} A_N = q-p$ and the sequence $(\sqrt[3]{n})_{n=1}^{\infty}$ is equidistributed mod 1.