FOURIER ANALYSIS. (Fall 2016)

3. EXERCISES (Fri 7.10, 10-12 in room C322)

1. (i) Let $N \in \mathbf{N}$. Show that there exists a non-trivial function $f \in L^1[-\pi,\pi]$ such that $F_N * f(x) = 0$ for all x.

(ii) Is there a non-trivial function $f \in L^1[-\pi,\pi]$ so that $F_N * f(x) = 0$ for all x and for *all* $N \geq 0$?

2. Use the results of lectures and verify that the Fourier-series of an integrable function converges at any point of differentiability of f.

3. Let $a > 0$ be a real number. Suppose the sequence $(x_n)_{n=1}^\infty$ is equidistributed (mod 1). Show that if $a \in \mathbf{Z} \setminus \{0\}$, then also the sequence $(ax_n)_{n=1}^\infty$ is equidistributed (mod 1).

Does the same result hold also for all $\alpha \notin \mathbf{Q}$?

4. Show that the sequence $(< a \log n >)_{n=1}^\infty$ is *not* equidistributed (mod 1) for any $a \in \mathbf{R}$.

5. Prove Corollary 4.8; that is, show that if a 2π-periodic function $f(x)$ is piecewise C^1, then its Fourier series converges at every point, and

$$\lim_{N\to\infty} S_N f(x) = \lim_{t\to 0} \frac{f(x+t)+f(x-t)}{2}, \qquad x \in [-\pi,\pi].$$

6. Let $f \in C_\#(-\pi,\pi)$. Assume that f has another period $\beta > 0$: $f(\beta + x) = f(x)$ for all x. Show that if f is constant if $\beta/2\pi$ is irrational.

7*[1] Is the sequence $(\sqrt[3]{n})_{n=1}^\infty$ equidistributed mod 1?

[1] These $*$-exercises are extras for afficinadoes, not required to get full points from exercises

Hints for some of the exercises:

T.4(ii): [Hint: Apply Weyl's criterium and compare the sum $\sum_{n=1}^{N} e^{2\pi i a \log n}$ with the corresponding integral.]

T.6: [Recall our result on the equidistribution of $\langle \alpha n \rangle$.]