FOURIER ANALYSIS. (fall 2016)

MODEL SOLUTIONS FOR SET 2

Exercise 1. (i) Show that if there exist the limit $A := \lim_{n\to\infty} a_n$, then also

$$\lim_{N\to\infty} \frac{a_0 + a_1 + \ldots + a_{n-1}}{N} = A$$

(ii) Use part (i) to verify that if the series $\sum_{n=0}^{\infty} b_n$ converges and has sum S, then it is also Cesaro summable, i.e. if $s_n := \sum_{k=0}^{n} b_n$, we have

$$S = \lim_{N\to\infty} \frac{s_0 + s_1 + \ldots + s_{n-1}}{N}.$$

Show by a counter example that the converse is not true.

(iii) Show that for Fourier series of given integrable function f the Fejer partial sum takes the form

$$\sigma_N f(x) = \sum_{n=-(N-1)}^{N-1} \Big(\frac{N-|n|}{N}\Big) \widehat{f}(n) e^{inx}.$$

Solution 1. (i) Suppose that the limit $A := \lim_{n\to\infty} a_n$ exists and fix $\varepsilon > 0$. Because the limit exists, we can find N_0 such that $|a_n - A| < \varepsilon$ for any $n > N_0$. For any $N > N_0$ we have

$$\begin{aligned}\left|\frac{a_0 + a_1 + \ldots + a_{N-1}}{N} - A\right| &\le \left|\frac{a_0 + a_1 + \ldots + a_{N_0} - N_0 A}{N}\right| \\ &+ \frac{|a_{N_0+1} - A| + \ldots + |a_{N-1} - A|}{N} \\ &\le \frac{|\sum_{n=0}^{N_0} a_n| + |A|(N_0+1)}{N} + \frac{(N - N_0 - 1)\varepsilon}{N} \\ &\le \frac{|\sum_{n=0}^{N_0} a_n| + |A|(N_0+1)}{N} + \varepsilon\end{aligned}$$

As the constant $C = |\sum_{n=0}^{N_0} a_n| + |A|(N_0+1)$ does not depend on N, we have that $C/N < \varepsilon$ for sufficiently large N. As ε was arbitrary, this proves that

$$\lim_{N\to\infty} \frac{a_0 + a_1 + \ldots + a_{N-1}}{N} = A$$

(ii) Suppose that the series $\sum_{n=0}^{\infty} b_n$ converges and has sum S. This means that if we denote $s_N = \sum_{n=0}^{N} b_n$, we have $S = \lim_{N\to\infty} s_N$. By applying (i), we obtain immediately

$$S = \lim_{N\to\infty} \frac{s_0 + s_1 + \ldots + s_{N-1}}{N}$$

As a counterexample, consider the sequence $b_n = (-1)^n$. The partial sums are $s_N = \sum_{n=0}^N b_n = (1 + (-1)^N)/2$. As all the partial sums are alternatingly 1 or 0, the series is not summable. However, it is Cesaro summable: for even N, we have

$$\frac{s_0 + \ldots + s_{N-1}}{N} = \frac{N/2}{N} = \frac{1}{2}$$

and for odd N

$$\frac{s_0 + \ldots + s_{N-1}}{N} = \frac{(N+1)/2}{N} = \frac{1}{2} + \frac{1}{2N} \to \frac{1}{2}$$

(iii) We can directly compute

$$\begin{aligned}\sigma_N f(x) &= \frac{1}{N}\sum_{k=0}^{N-1} S_k f(x) \\ &= \frac{1}{N}\sum_{k=0}^{N-1}\sum_{n=-k}^{k} \widehat{f}(n)e^{inx}\end{aligned}$$

Changing the order of summation gives

$$\begin{aligned}\sigma_N f(x) &= \frac{1}{N}\sum_{k=0}^{N-1}\sum_{n=-k}^{k} \widehat{f}(n)e^{inx} \\ &= \frac{1}{N}\sum_{n=-(N-1)}^{N-1}\sum_{k=|n|}^{N-1} \widehat{f}(n)e^{inx} \\ &= \sum_{n=-(N-1)}^{N-1}\Big(\frac{N-|n|}{N}\Big)\widehat{f}(n)e^{inx}\end{aligned}$$

Exercise 2. Show that Theorem 3.15 of lectures does not hold if $p = \infty$, i.e. there is $f \in L^\infty(-\pi,\pi)$ such that $\|f - \sigma_N f\|_{L^\infty(-\pi,\pi)} \not\to 0$ as $N \to \infty$.

Solution 2. We choose f to be the sign function, in other words,

$$f(x) = \begin{cases} 1, & \text{if } 0 < x < \pi \\ 0, & \text{if } x = 0 \\ -1, & \text{if } -\pi < x < 0 \end{cases}$$

Now we know that the trigonometric polynomial $\sigma_N f$ is continuous for any N. We will show that for any continuous function $g \in C(-\pi,\pi)$, we have $\|f - g\|_{L^\infty(-\pi,\pi)} \geq 1/2$: this would prove the claim.

Let g be a continuous function $g \in C(-\pi,\pi)$. As g is continuous at 0, we can find $0 < \delta < \pi$ such that

$$|g(x) - g(0)| < 1/2 \qquad \text{whenever } |x| < \delta.$$

First assume $g(0) \geq 0$ Now for any $-\delta < x < 0$ we can apply triangle inequality to get

$$|g(x) - f(x)| = |g(x) - g(0) + g(0) + 1| \geq |g(0) + 1| - |g(x) - g(0)| \geq |g(0) + 1| - 1/2 \geq 1/2.$$

As the set $(-\delta, 0)$ has measure $\delta > 0$, we get $\|f - g\|_{L^\infty(-\pi,\pi)} \geq 1/2$. Similarly, if $g(0) < 0$, for $0 < x < \delta$ we have

$$|g(x) - f(x)| = |g(x) - g(0) + g(0) - 1| \geq |g(0) - 1| - |g(x) - g(0)| \geq |g(0) - 1| - 1/2 \geq 1/2$$

and the set $(0, \delta)$ has measure $\delta > 0$.

Remark. We can actually show that the space of continuous periodic functions $C_\#(-\pi, \pi)$ (or, to be precise, the space of their equivalence classes) is a closed subspace $L^\infty(-\pi, \pi)$.

Remark. Additionally, for any measurable set $A \in [-\pi, \pi)$, we could use the characteristic function of A, denoted by χ_A, as a counterexample if the Lebesgue measure $m(A)$ of A satisfies $0 < m(A) < 2\pi$. To see this, we consider following disjoint sets, analogous to the topological interior, exterior and boundary:

$$int_m(A) = \{x \in [-\pi, \pi) : x \text{ has an open neighbourhood } V \text{ with } m(V \setminus A) = 0\}$$

$$ext_m(A) = \{x \in [-\pi, \pi) : x \text{ has an open neighbourhood } V \text{ with } m(V \cap A) = 0\}$$

$$\partial_m(A) = \{x \in [-\pi, \pi) : \text{for all open neighbourhoods } V \text{ of } x \text{ we have } 0 < m(V \cap A) < m(V)\}$$

As both $int_m(A)$ and $ext_m(A)$ are open and $[-\pi, \pi)$ is connected, we know that $\partial_m(A)$ can be empty only if either $int_m(A)$ or $ext_m(A)$ is the whole $[-\pi, \pi)$. But if $int_m(A) = [-\pi, \pi)$, then we see that $m(A) = 2\pi$ against the assumption on measure of A. A similar contradiction follows for $ext_m(A) = [-\pi, \pi)$. We may therefore pick a point $x \in \partial_m(A)$ and do the same argument as before.

Exercise 3. (i) Assume that $f \in L^1(-\pi, \pi)$ is odd i.e. $f(-x) = -f(x)$. Show that then the Fourier series of f is a pure sine series, i.e. can be expressed in terms of functions $\sin(nx)$, $n \in \mathbb{Z}$.

(ii) Conversely, if the Fourier series of $f \in L^1(-\pi, \pi)$ can be written as a sine series, deduce that $f(-x) = -f(x)$ almost surely for all $x \in (-\pi, \pi)$.

Solution 3. (i) Suppose f is odd. We use a substitution $t = -x$ compute that

$$\widehat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-inx} f(x) dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{int} f(-t) dt = -\frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-i(-nt)} f(t) dt = -\widehat{f}(-n).$$

This computation also shows that $\widehat{f}(0) = -\widehat{f}(0)$, so $\widehat{f}(0) = 0$. Let us now show that the Fourier series of f consists only of sine functions. As

$$\sin(x) = \frac{1}{2i} \left(e^{ix} - e^{-ix} \right)$$

we may represent the Fourier series as

$$\begin{aligned}\sum_{n=-\infty}^{\infty} \widehat{f}(n)e^{inx} &= \sum_{n=1}^{\infty} \left(\widehat{f}(n)e^{inx} + \widehat{f}(-n)e^{-inx}\right) \\ &= \sum_{n=1}^{\infty} \widehat{f}(n)\left(e^{inx} - e^{-inx}\right) \\ &= \sum_{n=1}^{\infty} 2i\widehat{f}(n)\sin(nx)\end{aligned}$$

(ii) We see that if the Fourier series of f can be written as a sine series, then

$$\begin{aligned}f(x) &\sim \sum_{n=1}^{\infty} a_n \sin(nx) \\ &= \sum_{n=1}^{\infty} \frac{a_n}{2i}(e^{inx} - e^{-inx}) \\ &= \sum_{n=1}^{\infty} \frac{a_n}{2i}e^{inx} - \sum_{n=-\infty}^{-1} \frac{a_{-n}}{2i}e^{inx}\end{aligned}$$

This means that $\widehat{f}(-n) = -\widehat{f}(n)$ for any integer n.

Define a function g by setting $g(x) = -f(-x)$. Then $g \in L^1(-\pi, \pi)$ and we can find its Fourier coefficients as

$$\begin{aligned}\widehat{g}(n) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} g(x)e^{-inx}\,dx \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} -f(-x)e^{-inx}\,dx \\ &= -\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{inx}\,dx \\ &= -\widehat{f}(-n) = \widehat{f}(n)\end{aligned}$$

Now we have obtained that $\widehat{g}(n) = \widehat{f}(n)$ for any integer n. By Theorem 3.16, we have $f(x) = g(x)$ for almost every $x \in (-\pi, \pi)$, so $f(-x) = -f(x)$ for almost every $x \in (-\pi, \pi)$.

Exercise 4. Define $f : [-\pi, \pi) \to \mathbb{R}$ by setting $f(x) = \cos(x/2)$. Compute the Fourier series of f. Does the Fourier series of f converge at every point ? Does it converge at zero? If so, what identity do you get by substituting $x = 0$?

Solution 4. We compute the Fourier coefficients:

$$\begin{aligned}
\widehat{f}(n) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-inx}\cos(x/2)\,dx \\
&= \frac{1}{4\pi}\int_{-\pi}^{\pi} e^{-inx}\left(e^{ix/2}+e^{-ix/2}\right)\,dx \\
&= \frac{1}{4\pi}\int_{-\pi}^{\pi}\left(e^{ix(1/2-n)}+e^{-ix(1/2+n)}\right)\,dx \\
&= \frac{1}{4\pi}\Big/_{-\pi}^{\pi}\left(\frac{1}{i(1/2-n)}e^{ix(1/2-n)}+\frac{1}{-i(1/2+n)}ee^{-ix(1/2+n)}\right) \\
&= \frac{1}{2\pi i(1-2n)}\left(e^{i\pi(1/2-n)}-e^{-i\pi(1/2-n)}\right)+\frac{1}{-2\pi i(1+2n)}\left(e^{-i\pi(1/2+n)}-e^{i\pi(1/2+n)}\right) \\
&= \frac{1}{2\pi i(1-2n)}\left(ie^{-i\pi n}+ie^{i\pi n}\right)+\frac{1}{2\pi i(1+2n)}\left(ie^{-i\pi n}+e^{i\pi n}\right) \\
&= \frac{1}{2\pi i(1-2n)}2i(-1)^n+\frac{1}{2\pi i(1+2n)}2i(-1)^n \\
&= \frac{(-1)^n}{\pi}\left(\frac{1}{1-2n}+\frac{1}{1+2n}\right) \\
&= \frac{2(-1)^n}{\pi(1-4n^2)}
\end{aligned}$$

These Fourier coefficients converge quickly enough to zero as $n\to\infty$ to make the Fourier series of f absolutely summable. Since f is also continuous, we can deduce again by Theorem 2.8 that the Fourier series converges uniformly to f. At $x=0$ we have the identity

$$\sum_{n=-\infty}^{\infty}\frac{2(-1)^n}{\pi(1-4n^2)}=\cos(0/2)=1.$$

As a curiosity, one could also deduce from this that

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{1-4n^2}=\frac{2+\pi}{4}.$$

Exercise 5. Define $f(x)=0$ for $x\in[-\pi,0]$, $f(x)=\pi-x$ for $x\in[0,\pi)$, and extend f to 2π-perodic function. Compute the Fourier series of f. In which points does the Fourier series of the function $f(x)$ converge and to what value?

Solution 5. We compute the Fourier coefficients. First, if $n = 0$, we have

$$\begin{aligned}
\widehat{f}(0) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx \\
&= \frac{1}{2\pi}\int_{0}^{\pi} (\pi - x)\,dx \\
&= \frac{\pi}{2} - \frac{1}{2\pi}\int_{0}^{\pi} x\,dx \\
&= \frac{\pi}{4}
\end{aligned}$$

For $n \neq 0$, we can use integration by parts:

$$\begin{aligned}
\widehat{f}(n) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\,dx \\
&= \frac{1}{2\pi}\int_{0}^{\pi} (\pi - x)e^{-inx}\,dx \\
&= \frac{1}{2}\int_{0}^{\pi} e^{-inx}\,dx - \frac{1}{2\pi}\int_{0}^{\pi} xe^{-inx}\,dx \\
&= \frac{1}{-2in}(e^{-in\pi} - 1) - \frac{1}{2\pi}\left(\pi\frac{1}{-in}e^{-in\pi} - \int_{0}^{\pi}\frac{1}{-in}e^{-inx}\,dx\right) \\
&= \frac{i}{2n}((-1)^n - 1) - \frac{i}{2n}(-1)^n + \frac{1}{2\pi}\int_{0}^{\pi}\frac{1}{-in}e^{-inx}\,dx \\
&= \frac{-i}{2n} + \frac{-1}{2\pi n^2}\left(e^{-in\pi} - 1\right) \\
&= \frac{-i}{2n} + \frac{1 - (-1)^n}{2\pi n^2}
\end{aligned}$$

The simplest way to see that the Fourier series converges everywhere is by using Dini's criterion. We define function $g : [-\pi, \pi] \to \mathbb{C}$ by setting $g(0) = \pi/2$ and $g(x) = f(x)$ for any $x \neq 0$. As g and f coincide almost everywhere, they have the same Fourier coefficients. After extending g to be 2π-periodic, we will show that for any $x_0 \in [-\pi, \pi]$, Dini's criterion holds at x_0 so the Fourier series converges to $g(x_0)$.

For $-\pi < x_0 < 0$, write $\delta = \min(x_0 + \pi, -x_0)$. We have $g(x_0 + t) = 0$ whenever $|t| < \delta$, so we get

$$\int_0^{\pi}\left|\frac{g(x_0+t)+g(x_0-t)}{2} - g(x_0)\right|\frac{dt}{t} \leq \frac{1}{\delta}\int_{\delta}^{\pi}\left|\frac{g(x_0+t)+g(x_0-t)}{2} - g(x_0)\right|dt < \infty$$

For $0 < x_0 < \pi$ write $\delta = \min(\pi - x_0, x_0)$. We have $g(x_0 + t) = \pi - (x_0 + t)$ whenever $|t| < \delta$, so we get

$$\int_0^\pi \left|\frac{g(x_0+t)+g(x_0-t)}{2} - g(x_0)\right| \frac{dt}{t} \leq \int_0^\delta \left|\frac{\pi-(x_0+t)+\pi-(x_0-t)}{2} - (\pi - x_0)\right| \frac{dt}{t} + \frac{1}{\delta}\int_\delta^\pi \left|\frac{g(x_0+t)+g(x_0-t)}{2} - g(x_0)\right| dt < \infty$$

For $x_0 = 0$ we have

$$\int_0^\pi \left|\frac{g(t)+g(-t)}{2} - g(0)\right| \frac{dt}{t} = \int_0^\pi \left|\frac{\pi - t}{2} - \frac{\pi}{2}\right| \frac{dt}{t} = \int_0^\pi \frac{1}{2}\, dt = \frac{\pi}{2} < \infty$$

For $x_0 = \pi$ we have

$$\int_0^\pi \left|\frac{g(\pi+t)+g(\pi-t)}{2} - g(\pi)\right| \frac{dt}{t} = \int_0^\pi \left|\frac{\pi-(\pi-t)}{2}\right| \frac{dt}{t} = \int_0^\pi \frac{1}{2}\, dt = \frac{\pi}{2} < \infty$$

We have obtained that the Fourier series of f converges everywhere on the interval $[-\pi, \pi]$. For $x \neq 0$ it converges to $f(x)$ and it converges to $\frac{\pi}{2}$ at 0.

Exercise 6. Let $(K_n)_{n\geq 1}$ be a good sequence of kernels on the interval $(-\pi, \pi)$ (especially, the functions K_n are 2π-periodic). Prove in detail Theorem 3.10 in case $p = 1$, or in other words, that for every $g \in L^1(-\pi, \pi)$ it holds that

$$\lim_{n\to\infty} \|g - K_n * g\|_{L^1(-\pi,\pi)} = 0.$$

Solution 6. Let $(K_n)_{n\geq 1}$ be a good family of kernels, and $g \in L^1(-\pi, \pi)$. We first write

$$\begin{aligned}
\|g - K_n * g\|_{L^1(-\pi,\pi)} &= \int_{-\pi}^{\pi} |g(x) - (K_n * g)(x)|\, dx \\
&= \int_{-\pi}^{\pi} \left| g(x) - \frac{1}{2\pi}\int_{-\pi}^{\pi} K_n(y) g(x-y)\, dy \right| dx \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \left| \int_{-\pi}^{\pi} g(x) - K_n(y) g(x-y)\, dy \right| dx
\end{aligned}$$

Using the fact that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} K_n(y)\, dy = 1$$

we may write

$$\begin{aligned}\|g - K_n * g\|_{L^1(-\pi,\pi)} &\leq \frac{1}{2\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} |g(x) - g(x-y)||K_n(y)|\,dy\,dx \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} |K_n(y)| \int_{-\pi}^{\pi} |g(x) - g(x-y)|\,dx\,dy.\end{aligned}$$

We know from real analysis that

$$\lim_{y\to 0}\int_{-\pi}^{\pi} |g(x) - g(x-y)|\,dx = 0.$$

Let $\varepsilon > 0$ be arbitrary. Choose $\delta > 0$ such that

$$\int_{-\pi}^{\pi} |g(x) - g(x-y)|\,dx < \varepsilon$$

for all $y \in (-\delta, \delta)$. Applying triangle inequality we get the estimate

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |g(x) - g(x-y)|\,dx \leq 2\|g\|_{L^1(-\pi,\pi)}.$$

Combining these estimates with the calculations above gives

$$\begin{aligned}\|g - K_n * g\|_{L^1(-\pi,\pi)} &\leq \frac{1}{2\pi}\int_{-\delta}^{\delta} |K_n(y)| \int_{-\pi}^{\pi} |g(x) - g(x-y)|\,dx\,dy \\ &\quad + \frac{1}{2\pi}\int_{|y|>\delta} |K_n(y)| \int_{-\pi}^{\pi} |g(x) - g(x-y)|\,dx\,dy \\ &\leq \frac{1}{2\pi}\int_{-\delta}^{\delta} |K_n(y)|\varepsilon\,dy + 2\|g\|_{L^1(-\pi,\pi)}\int_{|y|>\delta} |K_n(y)|\,dy.\end{aligned}$$

Now because K_n is a good sequence of kernels,

$$\varepsilon\frac{1}{2\pi}\int_{-\delta}^{\delta} |K_n(y)|\,dy \leq C\varepsilon,$$

for some constant $C > 0$. Addiotionally for all sufficiently large n we have

$$2\|g\|_{L^1}\int_{|y|>\delta} |K_n(y)|\,dy \leq \varepsilon.$$

So we have

$$\|g - K_n * g\|_{L^1(-\pi,\pi)} \leq (C+1)\varepsilon$$

for large enough n. Because $\varepsilon > 0$ is arbitrary, the claim follows.

Exercise 7*. Use the results of lectures so far to prove rigorously that every function $f : [0,\pi] \to \mathbb{C}$ that is Hölder-continuous (i.e. $|f(x) - f(y)| \leq C|x-y|^\alpha$ for some $\alpha \in (0,1]$) and satisfies $f(0) = f(\pi) = 0$ can at each point $x \in [0,\pi]$ be expressed as a convergent sine series

$$f(x) = \sum_{k=1}^{\infty} c_k \sin(kx).$$

Find an expression for the coefficients of c_k.

Solution 7*. Let us continue f on the interval $[-\pi,\pi]$ by setting $f(x) = -f(-x)$ when $-\pi \leq x \leq 0$, so f is an odd function. Now by Exercise 3 we can represent the Fourier series of f as a sine series

$$f(x) \sim \sum_{n=1}^{\infty} 2i\widehat{f}(n)\sin(nx).$$

The coeffients of the sine series are given by

$$\begin{aligned} c_n = 2i\widehat{f}(n) &= \frac{i}{\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\,dx \\ &= \frac{i}{\pi}\int_{-\pi}^{0} f(x)e^{-inx}\,dx + \frac{i}{\pi}\int_{0}^{\pi} f(x)e^{-inx}\,dx \\ &= \frac{i}{\pi}\int_{0}^{\pi} f(x)(e^{-inx} - e^{inx})\,dx = \frac{2}{\pi}\int_0^\pi f(x)\sin(nx)\,dx. \end{aligned}$$

We now consider the convergence of the Fourier series. We apply Dini's criterion. For $0 < x_0 < \pi$, writing $\delta = \min(x, \pi - x)$ we have

$$\begin{aligned} \int_0^\pi \left| \frac{f(x_0+t) + f(x_0-t)}{2} - f(x_0)\right| \frac{dt}{t} &= \int_0^\delta \left| \frac{f(x_0+t) + f(x_0-t)}{2} - f(x_0)\right| \frac{dt}{t} \\ &\quad + \int_\delta^\pi \left| \frac{f(x_0+t) + f(x_0-t)}{2} - f(x_0)\right| \frac{dt}{t} \\ &\leq \int_0^\delta \frac{|f(x_0+t) - f(x_0)| + |f(x_0-t) - f(x_0)|}{2}\frac{dt}{t} \\ &\quad + \frac{1}{\delta}\int_\delta^\pi \frac{|f(x_0+t)| + |f(x_0-t)|}{2} + |f(x_0)|\,dt \\ &\leq \int_0^\delta Ct^{\alpha-1}\,dt + \frac{1}{\delta}\left(\pi|f(x_0)| + 2\int_0^\pi |f(t)|\,dt\right) < \infty. \end{aligned}$$

This proves that the Fourier series converges to $f(x_0)$. As $\sin(n\pi) = 0$ for any integer n, we have shown that f can be represented as a convergent sine series

$$f(x) = \sum_{k=1}^{\infty} c_k \sin(kx).$$

with coefficients

$$c_n = \frac{2}{\pi} \int_0^\pi f(x) \sin(nx)\, dx$$