FOURIER ANALYSIS. (Fall 2016)

2. EXERCISES (Fri 30.09, 10-12 in room C322)

1. **(i)** Show that if there exist the limit $A := \lim_{n\to\infty} a_n$, then also

$$\lim_{N\to\infty} \frac{a_0 + a_1 + \ldots + a_{n-1}}{N} = A$$

(ii) Use part (i) to verify that if the series $\sum_{n=0}^{\infty} b_n$ converges and has sum S, then it is also Cesaro summable, i.e. if $s_n := \sum_{k=0}^{n} b_n$, we have

$$S = \lim_{N\to\infty} \frac{s_0 + s_1 + \ldots + s_{n-1}}{N}.$$

Show by a counter example that the converse is not true.

(ii) Show that for Fourier series of given integrable function f the Fejer partial sum takes the form

$$\sigma_N f(x) = \sum_{n=-(N-1)}^{N-1} \Big(\frac{N-|n|}{N}\Big) \widehat{f}(n) e^{inx}.$$

2. Show that Theorem 3.15 of lectures does not hold if $p = \infty$, i.e. there is $f \in L^\infty(-\pi,\pi)$ such that $\|f - \sigma_N f\|_{L^\infty(-\pi,\pi)} \not\to 0$ as $N \to \infty$.

3. **(i)** Assume that $f \in L^1(-\pi,\pi)$ is odd i.e. $f(-x) = -f(x)$. Show that then the Fourier series of f is a pure sine series, i.e. can be expressed in terms of functions $\sin(nx)$, $n \in \mathbf{Z}$.

(ii) Conversely, if the Fourier series of $f \in L^1(-\pi,\pi)$ can be written as a sine series, deduce that $f(-x) = -f(x)$ almost surely for all $x \in (-\pi,\pi)$.

4. Define $f : [-\pi,\pi) \to \mathbf{R}$ by setting $f(x) = \cos(x/2)$. Compute the Fourier series of f. Does the Fourier series of f converge at every point ? Does it converge at zero? If so, what identity do you get by substituting $x = 0$?

5. Define $f(x) = 0$ for $x \in [-\pi, 0]$, $f(x) = \pi - x$ for $x \in [0,\pi)$, and extend f to 2π-perodic function. Compute the Fourier series of f. In which points does the Fourier series of the function $f(x)$ converge and to what value?

6. Let $(K_n)_{n\geq 1}$ be a good sequence of kernels on the interval $(-\pi,\pi)$ (especially, the functions K_n are 2π-periodic). Prove in detail Theorem 3.10 in case $p = 1$, or in other words, that for every $g \in L^1(-\pi,\pi)$ it holds that

$$\lim_{n\to\infty} \|g - K_n * g\|_{L^1(-\pi,\pi)} = 0.$$

7[*1] Use the results of lectures so far to prove rigorously that every function $f : [0, \pi] \to \mathbf{C}$ that is Hölder-continuous (i.e. $|f(x) - f(y)| \leq C|x - y|^\alpha$ for some $\alpha \in (0, 1]$) and satisfies $f(0) = f(\pi) = 0$ can at each point $x \in [0, \pi]$ be expressed as a convergent sine series

$$f(x) = \sum_{k=1}^{\infty} c_k \sin(kx).$$

Find an expression for the coefficients of c_k.

[1]These $*$-exercises are extras for afficinadoes, not required to get full points from exercises

Hints for some of the exercises:

T.4(ii): [Write $\cos(x/2)$ with the aid of Euler's formula.]

T.6: [Follow the idea of the proof of Theorem 3.10 – in case $p = 1$ you can avoid completely the use of Hölder's inequality.]