FOURIER ANALYSIS. (fall 2016)

MODEL SOLUTIONS FOR SET 1

Exercise 1. Compute the Fourier coefficients of the function $f(x) = \pi - |x|$, for $|x| \leq \pi$.

Solution 1. First of all,

$$\widehat{f}(0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} (\pi - |x|)\ dx = \pi - \frac{2}{2\pi} \int_{0}^{\pi} x\, dx = \pi/2.$$

Let us now compute $\widehat{f}(n)$ for $n \neq 0$.

$$\begin{aligned}
\widehat{f}(n) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-inx} (\pi - |x|)\ dx \\
&= -\frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-inx} |x|\, dx \\
&= -\frac{1}{2\pi} \int_{0}^{\pi} \left(e^{-inx} + e^{inx}\right) x\, dx \\
&= -\frac{1}{\pi} \int_{0}^{\pi} \cos(nx) x\, dx
\end{aligned}$$

The above integral can be computed by standard integration by parts. The antiderivative is

$$\int x \cos(nx)\, dx = x\frac{\sin(nx)}{n} - \int \frac{\sin(nx)}{n}\, dx = x\frac{\sin(nx)}{n} - \frac{-\cos(nx)}{n^2} = x\frac{\sin(nx)}{n} + \frac{\cos(nx)}{n^2}$$

Thus

$$-\frac{1}{\pi} \int_{0}^{\pi} \cos(nx) x\, dx = \frac{1}{\pi} \frac{-\pi n \sin(\pi n) + 1 - \cos(\pi n)}{n^2}$$

For integer $n \neq 0$, we have

$$\frac{1}{\pi} \frac{-\pi n \sin(\pi n) + 1 - \cos(\pi n)}{n^2} = \frac{1 - \cos(\pi n)}{\pi n^2} = \begin{cases} 0, & \text{if } n \text{ is even} \\ \frac{2}{\pi n^2}, & \text{if } n \text{ is odd} \end{cases}$$

So we have

$$\widehat{f}(n) = \begin{cases} \frac{\pi}{2}, & \text{if } n = 0 \\ 0, & \text{if } n \neq 0 \text{ is even} \\ \frac{2}{\pi n^2}, & \text{if } n \text{ is odd} \end{cases}$$

Exercise 2. Compute the Fourier coefficients of the function $f(x) = e^x$, $x \in [-\pi, \pi]$.

Solution 2. From the definition of the Fourier coefficients we can compute

$$\begin{aligned}
\widehat{f}(n) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{x-inx}\,dx \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{(1-in)x}\,dx \\
&= \frac{1}{2\pi}\frac{1}{1-in}\left(e^{(1-in)\pi}-e^{-(1-in)\pi}\right) \\
&= \frac{1}{2\pi}\frac{1}{1-in}\left(e^{\pi}e^{-in\pi}-e^{-\pi}e^{in\pi}\right) \\
&= \frac{(-1)^n\left(e^{\pi}-e^{-\pi}\right)}{2\pi(1-in)} \\
&= \frac{(-1)^n\sinh(\pi)}{\pi(1-in)}
\end{aligned}$$

Exercise 3. Assume that $f \in L^1(-\pi,\pi)$ is even, i.e. $f(-x) = f(x)$. Show that then the Fourier series of f is a pure cosine series, i.e. can be expressed in terms of functions $\cos(nx)$, $n \in \mathbb{Z}$.

Solution 3. Suppose f is even. We use a substitution $t = -x$ compute that

$$\widehat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-inx}f(x)dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{int}f(-t)dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-i(-nt)}f(t)dt = \widehat{f}(-n).$$

Let us now show that the Fourier series of f consists only of cosine functions. As

$$\cos(x) = \frac{1}{2}\left(e^{ix}+e^{-ix}\right)$$

we may represent the Fourier series as

$$\begin{aligned}
\sum_{n=-\infty}^{\infty}\widehat{f}(n)e^{inx} &= \widehat{f}(0) + \sum_{n=1}^{\infty}\left(\widehat{f}(n)e^{inx}+\widehat{f}(-n)e^{-inx}\right) \\
&= \widehat{f}(0) + \sum_{n=1}^{\infty}\widehat{f}(n)\left(e^{inx}+e^{-inx}\right) \\
&= \widehat{f}(0) + \sum_{n=1}^{\infty}2\widehat{f}(n)\cos(nx)
\end{aligned}$$

Exercise 4. How do you express the Fourier coefficients $\widehat{g}(n)$ assuming that you know those of f when f is 2π-periodic and

(i) $g(x) = f(x_0 + x)$?

(ii) $g(x) = f(2x)$, for $x \in [-\pi,\pi]$?

Solution 4. **(i)** Using subtitution $y = x + x_0$ shows that

$$\begin{aligned}\widehat{g}(n) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} g(x)e^{-inx}\,dx \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x+x_0)e^{-inx}\,dx \\ &= \frac{1}{2\pi}\int_{-\pi+x_0}^{\pi+x_0} f(y)e^{-in(y-x_0)}\,dy \\ &= e^{inx_0}\frac{1}{2\pi}\int_{-\pi+x_0}^{\pi+x_0} f(y)e^{-iny}\,dy\end{aligned}$$

Now, the function $y \to f(y)e^{-iny}$ is 2π-periodic, so its integral over any interval of length 2π does not depend on the endpoints of the interval. Thus

$$\begin{aligned}\widehat{g}(n) &= e^{inx_0}\frac{1}{2\pi}\int_{-\pi+x_0}^{\pi+x_0} f(y)e^{-iny}\,dy \\ &= e^{inx_0}\frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)e^{-iny}\,dy \\ &= e^{inx_0}\widehat{f}(n)\end{aligned}$$

(ii) We will prove that

$$\widehat{g}(n) = \begin{cases} \widehat{f}(n/2), & \text{if } n \text{ is even} \\ 0, & \text{if } n \text{ is odd} \end{cases}$$

We make a substitution $y = 2x$ and obtain

$$\begin{aligned}\widehat{g}(n) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(2x)e^{-inx}\,dx = \frac{1}{4\pi}\int_{-2\pi}^{2\pi} f(y)e^{-i(n/2)y}\,dy \\ &= \frac{1}{4\pi}\left(\int_{-2\pi}^{0} f(y)e^{-i(n/2)y}\,dy + \int_{0}^{2\pi} f(y)e^{-i(n/2)y}\right) \\ &= \frac{1}{4\pi}\int_{0}^{2\pi} f(y)e^{-i(n/2)y}\,dy(e^{-in\pi}+1)\end{aligned}$$

Now if n is even, then we find that

$$\frac{1}{4\pi}\int_{0}^{2\pi} f(y)e^{-i(n/2)y}\,dy(e^{-in\pi}+1) = \frac{1}{2\pi}\int_{0}^{2\pi} f(y)e^{-i(n/2)y}\,dy = \widehat{f}(n/2)$$

For odd n, we get

$$\frac{1}{4\pi}\int_{0}^{2\pi} f(y)e^{-i(n/2)y}\,dy(e^{-in\pi}+1) = \frac{1}{4\pi}\int_{0}^{2\pi} f(y)e^{-i(n/2)y}\,dy \cdot 0 = 0$$

Remark. We have in fact a more general result: if k is a positive integer and $g(x) = f(kx)$, then

$$\widehat{g}(n) = \begin{cases} \widehat{f}(n/k), & \text{if } k \text{ divides } n \\ 0, & \text{if } k \text{ does not divide } n \end{cases}$$

The essential observation when proving this is that $\sum_{j=0}^{p-1} e^{-i(j/p)\pi} = 0$ for any integer $p \geq 2$, the details are left for an interested reader.

Exercise 5. Let $a \in L^1(-\pi, \pi)$ be an integrable function with $\int_{-\pi}^{\pi} a(x) = 2\pi$. Assume that $a(x) = 0$ for $|x| \geq \pi$. Show that the (2π-periodifications) of the functions

$$k_n(x) = na(nx) \quad \text{for} \quad x \in [-\pi, \pi], \quad n \in \mathbb{Z}^+$$

give a good sequence of kernels.

Solution 5. We simply verify that all three conditions in Definition 3.4. are satisfied.

(3.4) For any n, we have

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} k_n(x)\, dx = 1$$

Let n be arbitrary. Making the change of variables $y = nx$ we get by using the fact $a(x) = 0$ for $|x| > \pi$ that

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} k_n(x)\, dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} na(nx)\, dx = \frac{1}{2\pi} \int_{-n\pi}^{n\pi} a(y)\, dy = \frac{1}{2\pi} \int_{-\pi}^{\pi} a(y)\, dy = 1$$

(3.5) For any n, we have for some constant C_0

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |k_n(x)|\, dx \leq C_0$$

Fix n. Making again the change of variables $y = nx$ and using the fact that $a \in L^1(-\pi, \pi)$, we get

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |k_n(x)|\, dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} |na(nx)|\, dx = \frac{1}{2\pi} \int_{-n\pi}^{n\pi} |a(y)|\, dy = \frac{1}{2\pi} \int_{-\pi}^{\pi} |a(y)|\, dy = \|a\|_1/2\pi$$

(3.6) For any $\delta > 0$, we have

$$\lim_{n\to\infty} \int_{\delta \leq |x| \leq \pi} |k_n(x)|\, dx = 0$$

Fix $\delta > 0$. Then for any $n > \pi/\delta$, we have

$$\int_{\delta \leq |x| \leq \pi} |k_n(x)|\, dx = \int_{\delta \leq |x| \leq \pi} |na(nx)|\, dx = \int_{\delta \leq |x| \leq \pi} 0\, dx = 0$$

We have shown that $\{k_n\}_{n=1}^{\infty}$ is a good sequence of kernels.

Exercise 6. According to lectures the Fourier coefficients $\widehat{f}(n)$ of $C^k_\#$-functions f tend to zero at least at the rate n^{-k} kun $|n| \geq 1$. Prove a partial converse to this result: : show that if $f \in C_\#(-\pi, \pi)$ and for each $k \in \mathbb{N}$ there is a constant $C = C_k$ such that

$$|\widehat{f}(n)| \leq C_k(1+|n|)^{-k} \quad \text{forevery} \;\; n \in \mathbb{Z},$$

then it holds that$f \in C^\infty_\# := \cap_{k\geq 1} C^k_\#(-\pi, \pi)$.

Solution 6. We will show that f is in C^1 and deduce the rest by induction. First of all, just from the case $k = 2$ we know that

$$|\widehat{f}(n)| \leq C_2(1+|n|)^{-2}$$

for some constant C_2. Thus the Fourier series of f converges absolutely, and hence by Theorem 2.8 the Fourier series converges uniformly to f on $[-\pi, \pi]$. We can hence write

$$f(x) = \sum_{n=-\infty}^{\infty} \widehat{f}(n) e^{inx}.$$

If we show that the series on the right hand side is continuously differentiable, then so is f. To do this we first define the function g by

$$g(x) = \sum_{n=-\infty}^{\infty} in\widehat{f}(n) e^{inx}.$$

Note that the series defining g is well-defined and converges uniformly to g since

$$|in\widehat{f}(n)| \leq C_3|n|(1+|n|)^{-3} \leq C_3(1+|n|)^{-2}$$

for another constant C_3. We want to justify saying that $f' = g$. For this, recall a result from Analysis II which says that if f_n is a sequence of functions converging to f at the point x and the sequence f'_n converges uniformly to another function g, then $f'(x) = g(x)$. Applying this result to the partial sums

$$f_N(x) = \sum_{n=-N}^{N} \widehat{f}(n) e^{inx} \quad , \quad f'_N(x) = \sum_{n=-N}^{N} in\widehat{f}(n) e^{inx}$$

shows that $f'(x) = g(x)$ for all x as wanted. Since g is a uniform limit of continuous functions, it is continuous. Thus f is in C^1. Moreover, for all k it holds that

$$|\widehat{g}(n)| = |in\widehat{f}(n)| \leq |n| C_{k+1}(1+|n|)^{-k+1} \leq C_{k+1}(1+|n|)^{-k}.$$

Hence $g = f'$ satisfies the same condition as the original function f. Thus by the same arguments as above g must also be in C^1, so f is in C^2. By induction we see that f has to be in C^k for all $k \geq 0$. We are done.

Exercise 7*. During the lectures the result of Exercise 1 was used to prove that

$$\sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} = \frac{\pi^2}{8}.$$

Use this fact to prove the famous Euler formula

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots = \frac{\pi^2}{6}.$$

Solution 7*. We define following three sequences:

$$s_m = \sum_{n=1}^{m} \frac{1}{n^2}$$

$$a_m = \sum_{n=1}^{m} \frac{1}{(2n-1)^2}$$

$$b_m = \sum_{n=1}^{m} \frac{1}{2^{2(n-1)}}$$

We can determine by integral test that

$$s_m \leq 1 + \int_1^m \frac{1}{x^2}\,dx = 2 - \frac{1}{m} \leq 2$$

so the sequence s_m is increasing and bounded; in particular, the limit $S = \lim_{m\to\infty} s_m$ exists. Next we show that

$$s_m \leq a_m b_m \leq S$$

We use the fact that we can write any positive integer n as $n = 2^k r$ for some non-negative integer k and odd integer r. If we multiply open the expression $a_m b_m$, we see that

$$a_m b_m = \sum_{n=2^k r, k<m, r\leq 2m-1} \frac{1}{n^2}$$

As all the integers n we some over are distinct, we immediately get that

$$a_m b_m \leq S$$

For the other inequality, we observe that for any $2^k r \leq m$ we have $r \leq m \leq 2m-1$, and $2^k \leq m$ so $k < m$. This gives

$$a_m b_m \geq s_m$$

Now we see that

$$S = \lim_{m\to\infty} s_m \leq \lim_{m\to\infty} a_m b_m \leq S$$

But we already know that

$$\lim_{m\to\infty} a_m = \frac{\pi^2}{8}$$

We can also find by the geometric series formula that

$$\lim_{m\to\infty} b_m = \frac{1}{1-\frac{1}{4}} = \frac{4}{3}$$

Combining this, we see that

$$S = \lim_{m\to\infty} a_m b_m = \frac{\pi^2}{8} \cdot \frac{4}{3} = \frac{\pi^2}{6}$$