FOURIER ANALYSIS. (fall 2016)

1. EXERCISES (pe 23.09, 10-12 in room C322)

1. Compute the Fourier coefficients of the function $f(x) = \pi - |x|$, for $|x| \leq \pi$.

2. Compute the Fourier coefficients of the function $f(x) = e^x$, $x \in [-\pi, \pi]$.

3. Assume that $f \in L^1(-\pi, \pi)$ is even, i.e. $f(-x) = f(x)$. Show that then the Fourier series of f is a pure cosine series, i.e. can be expressed in terms of functions $\cos(nx)$, $n \in \mathbf{Z}$.

4. How do you express the Fourier coefficients $\widehat{g}(n)$ assuming that you know those of f when f is 2π-periodic and

(i) $g(x) = f(x_0 + x)$?

(ii) $g(x) = f(2x)$ for $x \in [-\pi, \pi]$?

5. Let $a \in L^1(-\pi, \pi)$ be an integrable function with $\int_{-\pi}^{\pi} a(x) = 2\pi$. Assume that $a(x) = 0$ for $|x| \geq \pi$. Show that the (2π-periodifications) of the functions

$$k_n(x) = na(nx) \quad \text{for} \quad x \in [-\pi, \pi], \quad n \in \mathbf{Z}^+$$

give a good sequence of kernels.

6. According to lectures the Fourier coefficients $\widehat{f}(n)$ of $C^k_\#$-functions f tend to zero at least at the rate n^{-k} kun $|n| \geq 1$. Prove a partial converse to this result: : show that if $f \in C_\#(-\pi, \pi)$ and for each $k \in \mathbf{N}$ there is a constant $C = C_k$ such that

$$|\widehat{f}(n)| \leq C_k(1 + |n|)^{-k} \quad \text{forevery} \ \ n \in \mathbf{Z},$$

then it holds that$f \in C^\infty_\# := \cap_{k \geq 1} C^k_\#(-\pi, \pi)$.

7*[1] During the lectures the result of Exercise 1 was used to prove that

$$\sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} = \frac{\pi^2}{8}.$$

Use this fact to prove the famous Euler formula

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \; = \; 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots \; = \; \frac{\pi^2}{6}.$$

[1] These ∗-exercises are extras for afficinadoes, not required to get full points from exercises

Hints for some of the exercises:

T.4(ii): [Guess the answer by writing formally f as its Fourier series and substitute $x = 2x$]

T.6: [Use results from the analysis courses to verify that in this situation you may differentiate the]

T.7: [Note first what happens to the first series if you multiply it by 2^{-2k}: you get sum of reciprocals of squares of certain numbers.]