Finite Model Theory: Lecture 1

Jonni Virtema

University of Helsinki, Finland
jonni.virtema@gmail.com

19th of September, 2016

Relations

- 1-ary relations over X are just subsets of X.
- 2-ary relations over X are sets of pairs (x, y) s.t $x, y \in X$.
- 3-ary relations over X are sets of triplets (x, y, z) s.t $x, y, z \in X$.
- etc.

Definition (n-ary relation)

Let X be a set and $n \in \mathbb{Z}_+$ a positive natural number. An *n-ary relation over X* is a subset of the n-fold cartesian product

$$X^n = \{(x_0, \ldots, x_{n-1}) \mid x_0, \ldots, x_{n-1} \in X\}.$$

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Binary relations

- Already powerful and versitile
- Examples: Edge relation E in graphs, set inclusion $\in$ in set theory, order relation $\leq$, etc.
- Many interesting properties: reflexive, symmetric, transitive, etc
- Unary function $f : X \to X$ is in fact a binary relation $\{(x, f(x)) \mid x \in X\}$.

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Functions

- 1-ary functions over X map elements of X to elements of X, $f(a) = b$.
- 2-ary functions over X map pairs of elements of X to elements of X, $g(a, b) = c$.

Definition (n-ary function)

Let X be a set and $n \in \mathbb{Z}_+$ a positive natural number. An *n-ary function f over X* is a mapping $f : X^n \to X$.

Often functions are identified with their graphs. That is, an n-ary function $f : X^n \to X$ as an $n+1$-ary relation:

$$\{(x_0, \ldots, x_{n-1}, f(x_0, \ldots, x_{n-1})) \mid x_0, \ldots, x_{n-1} \in X\}$$

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More on functions

- By $\mathrm{dom}(f)$ and $\mathrm{rg}(f)$ we denote the domain and the range on a function f.
- $f : A \to B$ means that $A = \mathrm{dom}(f)$ and $\mathrm{rg}(f) \subseteq B$.
- Properties of functions: injection, surjection, bijection.
- Example: $+$ is a binary function on $\mathbb{N}$, $+(a, b) = a + b$, $+(2, 3) = 5$.

Symbols and vocabularies

- Symbols:
  - Relation symbols: R, S, R_1, R_2 etc. Relation symbol has an arity $\mathrm{ar}(R) \in \mathbb{Z}_+$.
  - Function symbols: f, g, f_1, f_2 etc. Function symbol has an arity $\mathrm{ar}(f) \in \mathbb{Z}_+$.
  - Constant symbols: c, d, c_1, c_2 etc. Constant symbol has an arity $\mathrm{ar}(c) = 0$.
- Symbols do not have any structure. Only type (relation/function/constant) and arity.
- Vocabulary is a set of symbols.
  - We use $\tau, \sigma, \tau_1, \tau_2$ etc. to denote vocabularies.
  - $\mathrm{Con}(\tau)$ denotes the set of constant symbols in τ.
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Model

Definition

Let τ be a vocabulary. A pair $\mathfrak{A} = (A, T)$ is called a τ-model, it the following conditions hold:

1. $A \neq \emptyset$; the set A is called as the *universe* or *domain* of $\mathfrak{A}$. We denote $\mathrm{Dom}(\mathfrak{A}) = A$.
2. T is a function with domain $\mathrm{dom}(T) = \tau$. T maps every symbols $X \in \tau$ to its *interpretation* $T(X)$ in $\mathfrak{A}$ (denoted also by $X^{\mathfrak{A}}$).
3. For each $c \in \mathrm{Con}(\tau)$ the intepretation $c^{\mathfrak{A}} \in \mathrm{Dom}(\mathfrak{A})$.
4. For each $R \in \mathrm{Rel}(\tau)$ the intepretation $R^{\mathfrak{A}}$ is an n-ary relation over $\mathrm{Dom}(\mathfrak{A})$, where $n = \mathrm{ar}(R)$.
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instead of (A, T). When extremely lazy we write simply

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