Supplementary lecture notes

Evolution and the Theory of Games

QUITTING STRATGIES

by

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When quitting the game is an option ...

1. Quitting strategies. Deliberately quitting the iterated prisoner's dilemma traditionally has not been considered a strategic option [1]. However, it is not at all unthinkable that under certain circumstances it may be better to quit a game than to continue it, especially when you are loosing. The "certain circumstances" here refer to the wider context in which the game is embedded: if you quit playing, will you start a new game with another player, or will you play another kind of game altogether and what game would that be, or was this game that you just quitted all there is?

Quitting can be built into any strategy simply by adding a "quitting rule". For example, consider the TFT strategy with the additional rule:

"Quit whenever you receive a low payoff (S or P)."

Of course, the game may be terminated for other (random) reasons, but that is already taken care of by the discounting factor $\delta \in (0, 1)$.

Let's see how this works out for two strategies allC* and allD*, which are like allC and allD but with the above quitting rule.

allC*×allC* gets immediately in a (C×C)-cycle. The payoff per round is R to both players, so neither of the two will deliberately quit. The expected number of rounds is $(1 - \delta)^{-1}$, so the overall payoff to both players is $R/(1 - \delta)$.

allC*×allD*gives the sucker's payoff S to the allC* player, who therefore quits. The allD* player gets the payoff T and would like to continue, but cannot because his co player quits.

allD*×allD* gives the payoff P to both players, who thereafter quit.

The overall payoff matrix thus becomes

[1] I found only one paper on this topic written by S. Moresi & S. C. Salop; the precise coordinates I still have to look up ...

	allC*	allD*
allC*	$\frac{R}{1-\delta}$	S
allD*	T	P

If an individual plays only one game in his life, then allD* is an ESS because $P > S$. But also allC* is an ESS if $R/(1-\delta) > T$, i.e., if $1 - R/T < \delta < 1$, or in yet other words, if the probability of entering the next round is sufficiently large. Note how this is different in the original allC versus allD (without the asterisk, i.e., without a quitting rule) where allC is never an ESS and allD is always an ESS.

2. The population dynamical context.

Now suppose that after termination of the PD game (either by deliberately quitting or by other random causes as modelled by the discounting facto δ) both players choose a new opponent randomly selected from the population at large to start a new game.

We then have to take into account that the expected number of round per game is different for different strategy combinations. For example, the allC*×allC* contest in the previous section lasts on average $(1-\delta)^{-1} > 1$ rounds, while all other combinations last exactly one round.

I do not think the players that have quitted after the first round will wait for the allC*×allC* pairs to finish their play, which may take a very long time, especially when δ is close to one. Instead I think that as soon as a game is over, the players that have quitted will immediately look for a new partner and start a new game. Anyways, this is what we shall assume.

As a consequence, the number of games played during a player's life-time will be different for different strategies. The question is how this affects the conclusion in the previous section about the evolutionary stability of the strategy allC*?

To answer that question, we have to embed the game in a wider population dynamical context. This is what we do in the next two sections.

3. The short time scale without births and deaths

Here is one possible way to embed a game (any game) with a quitting rule into a population dynamical model:

Consider the strategies S_1 and S_2. Then there are four kinds of contests: $\mathrm{S}_1 \times \mathrm{S}_1$, $\mathrm{S}_1 \times \mathrm{S}_2$, $\mathrm{S}_2 \times \mathrm{S}_1$ and $\mathrm{S}_2 \times \mathrm{S}_2$. In a general $\mathrm{S}_i \times \mathrm{S}_j$ contest we shall always associate S_i (i.e., the one listed first) with the row player and S_j (i.e., the one listed second) with the column player.

Let a_{ijk} denote the payoff to the S_i player during the k^{th} round of a ($\mathrm{S}_i \times \mathrm{S}_j$)-contest, and let k_{ij} denote the round after which either or both players would quit if the game were still on. As it take two players to play, necessarily $k_{ij} = k_{ji}$. (Note that it may happen that $k_{ij} = \infty$, as in an allC*×allC* contest where neither player intends to quit ever.) The total expected payoff to the S_i player accumulated over all rounds then is

$$a_{ij} = \sum_{k=1}^{k_{ij}} a_{ijk} \delta^{k-1}, \tag{1}$$

and so the payoff matrix $\mathbf{A}$ of a single game is given by

	S_1	S_2
S_1	a_{11}	a_{12}
S_i	a_{21}	a_{22}

We shall assume an infinitely large population of players, i.e., infinite in number of individuals but not in terms of population densities, which are finite. Then, after each round there will be “free individuals”, i.e., individuals that are no longer playing, because their game terminated at the end of the last round. Before the next round starts (we count time in the number of rounds $\tau = 1, 2, 3, \ldots$ since we started looking at the population), these free individuals are paired again, randomly and without replacement. Let s_i denote the population density of free S_i individuals. The probability of forming a $\mathrm{S}_i \times \mathrm{S}_j$ pair is then

$$\frac{s_i(\tau)}{s_1(\tau) + s_2(\tau)} \cdot \frac{s_j(\tau)}{s_1(\tau) + s_2(\tau)}, \tag{2}$$

and the population density of newly formed pairs is

$$\frac{s_1(t) + s_2(t)}{2}. \tag{3}$$

Consequently, the population density s_{ij1} of $\mathrm{S}_i \times \mathrm{S}_j$ contests in their first round is

$$s_{ij1}(t+1) = \frac{1}{2} \frac{s_i(t) s_j(t)}{s_1(t) + s_2(t)}, \tag{4}$$

and for the population density s_{ijk} of $\mathrm{S}_i \times \mathrm{S}_j$ contests that just start their k^{th} round we have

$$s_{ijk}(t+1) = \delta\, s_{ij(k-1)}(t) \quad (k = 1, \ldots, k_{ij}). \tag{5}$$

Ignoring births and deaths (which we assume take place on a different and much longer timescale) the population will reach a quasi equilibrium that satisfies

$$s_{ijk} = \frac{1}{2}\, \delta^{k-1} \frac{s_i s_j}{s_1 + s_2} \quad (k = 1, \ldots, k_{ij}). \tag{6}$$

At the quasi-equilibrium, the payoff per round to all S_i players together is

$$\sum_{j=1}^{2}\sum_{k=1}^{k_{ij}} a_{ijk}s_{ijk}, \tag{7}$$

which, using (6), can be written as

$$s_i \sum_{j=1}^{2} \frac{a_{ij}\,s_j}{s_1+s_2} \tag{8}$$

where the a_{ij} are the same as in equation (1).

4. The long time scale with births and deaths

We now turn to a slower time scale t where births and death can no longer be ignored. In terms of this slower time, the length of a single round of a game is only a very small $\varepsilon > 0$.

Suppose that S_1 players produce only S_1-offspring and S_2 players only S_2-offspring. Assume further that reproduction is proportional to the payoff received (with proportionality constant $\alpha > 0$) and that the *per capita* death rate $\mu(t)$ is strategy-independent. Then from (8) we get

$$s_i(t+\varepsilon) = s_i(t) + \varepsilon\,\alpha\,s_i \sum_{j=1}^{2} \frac{a_{ij}\,s_j}{s_1+s_2} - \varepsilon\,\mu(t)\,s_i(t) \tag{9}$$

for $i = 1,2$. Subtracting $s_i(t)$ from both sides, dividing by ε and letting $\varepsilon \to 0$, we get the differential equation

$$\dot{s}_i = \alpha\,s_i \sum_{j=1}^{2} \frac{a_{ij}\,s_j}{s_1+s_2} - \mu(t)\,s_i. \tag{10}$$

This equation can be rewritten in terms of relative frequencies

$$p_i := \frac{s_i}{s_1+s_2},$$

which gives the purely frequency-dependent equation

$$\dot{p}_i = p_i\left(\sum_{j=1}^{2} a_{i,j}\,p_j - \sum_{j_1=1}^{2}\sum_{j_2=1}^{2} p_{j_1}a_{j_1j_2}p_{j_2}\right) \tag{11}$$

for $i = 1,2$.

4. Back to payoff matrices

To address the question "*who can invade whom?*", let $p_1 = p$ represent the frequency of the (initially rare) invader strategy S_1 and $p_2 = 1 - p$ the frequency of the (initially common) resident strategy S_2, and rewrite (12) as

$$\dot{p} = p(1-p)\Big(a_{12} - a_{22} - p(a_{12} - a_{22} + a_{21} - a_{11})\Big). \tag{12}$$

The factor $p(1-p)$ is non-negative, and hence invasion (or not) depend on the sign of the remaining factor: if positive, then S_1 can invade, but if negative, then it cannot. This leads to the non-invadability conditions

(1) $a_{12} < a_{22}$ or

(2) $a_{12} = a_{22}$ and $a_{21} > a_{11}$.

Note that these are the common ESS conditions for S_2 being an ESS given the payoff matrix **A** of a single game, i.e.,

	S_1	S_2
S_1	a_{11}	a_{12}
S_i	a_{21}	a_{22}

in which we do not take into account that free individuals can pair-up again. So, the grand conclusion is that the random pairing-up of free individuals to start a new game while other games are still going on does not make any difference for the ESS calculations.