BRIEF INTRODUCTION TO FOURIER SERIES

Version 1, Samuli Siltanen, September 14, 2016

1. Fourier series, real formulation

Assume that the function $f : \mathbb{R} \to \mathbb{R}$ is 2π-periodic (in other words, satisfies $f(x) = f(x + \nu 2\pi)$ for any $\nu \in \mathbb{Z}$) and can be written in the form

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos(nx) + b_n \sin(nx) \right), \tag{1}$$

where $a_0, a_1, a_2, \ldots$ and $b_1, b_2, \ldots$ are real-valued coefficients.

Computationally it is very useful to consider approximations of functions and signals by truncated Fourier series

$$f(x) \approx a_0 + \sum_{n=1}^{N} \left(a_n \cos(nx) + b_n \sin(nx) \right). \tag{2}$$

Then the practical question is: given f, how to determine the coefficients $a_0, a_1, a_2, \ldots, a_N$ and $b_1, b_2, \ldots, b_N$? Let us derive formulas for them.

The constant coefficient a_0 is found as follows. Integrate both sides of (1) from 0 to 2π:

$$\begin{aligned} \int_0^{2\pi} f(x)dx &= a_0 \int_0^{2\pi} dx + \\ &\quad + \sum_{n=1}^{\infty} a_n \int_0^{2\pi} \cos(nx)dx + \\ &\quad + \sum_{n=1}^{\infty} b_n \int_0^{2\pi} \sin(nx)dx, \end{aligned} \tag{3}$$

where we assumed that the orders of infinite summing and integration can be interchanged. Now it is easy to check that $\int_0^{2\pi} \cos(nx)dx = 0$ and $\int_0^{2\pi} \sin(nx)dx = 0$ and $\int_0^{2\pi} dx = 2\pi$. Therefore,

$$a_0 = \frac{1}{2\pi} \int_0^{2\pi} f(x)dx, \tag{4}$$

which can be interpreted as the average value of the function f over the interval $[0, 2\pi]$.

Further, fix any integer $m \geq 1$ and multiply both sides of (1) by $\cos(mx)$. Integration from 0 to 2π gives

$$\begin{aligned}\int_0^{2\pi} f(x)\cos(mx)dx &= a_0\int_0^{2\pi}\cos(mx)dx + \\ &+\sum_{n=1}^{\infty} a_n\int_0^{2\pi}\cos(nx)\cos(mx)dx + \\ &+\sum_{n=1}^{\infty} b_n\int_0^{2\pi}\sin(nx)\cos(mx)dx. \end{aligned} \tag{5}$$

We already know that $\int_0^{2\pi}\cos(mx)dx = 0$, so the term containing a_0 in the right hand side of (5) vanishes. Clever use of trigonometric identities allows one to see that

$$\int_0^{2\pi}\sin(nx)\cos(mx)dx = 0 \quad \text{for all } n \geq 1, \tag{6}$$

and that

$$\int_0^{2\pi}\cos(nx)\cos(mx)dx = 0 \quad \text{for all } n \geq 1 \text{ with } n \neq m. \tag{7}$$

The checking of (6) and (7) is left as an exercise. So actually the only nonzero term in the right hand side of (5) is the one containing the coefficient a_m. Another exercise is to verify this identity:

$$\int_0^{2\pi}\cos(nx)\cos(nx)dx = \pi. \tag{8}$$

Therefore, substituting (8) into (5) gives

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos(nx)dx. \tag{9}$$

A similar derivation shows that

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin(nx)dx. \tag{10}$$

One might be tempted to ask: what kind of functions allow a representation of the form (1)? Or: in what sense does the right-hand sum converge in (2) as $N \to \infty$? Also: under what assumptions can the order of infinite summing and integration can be interchanged in the derivations of (3) and (5)? These are deep and interesting mathematical questions which will not be further discussed in this short note.

2. Fourier series, complex formulation

Parametrize the boundary of the unit circle as

$$\{(\cos\theta, \sin\theta) \,|\, 0 \le \theta < 2\pi\}.$$

We will use the *Fourier basis functions*

$$\varphi_n(\theta) = (2\pi)^{-1/2} e^{in\theta}, \quad n \in \mathbb{Z}. \tag{11}$$

We can approximate 2π-periodic functions $f : \mathbb{R} \to \mathbb{R}$ following the lead of the great applied mathematician Joseph Fourier (1768–1830). Define cosine series coefficients using the L^2 inner product

$$\widehat{f}_n := \langle f, \varphi_n \rangle = \int_0^{2\pi} f(\theta)\, \overline{\varphi_n(\theta)}\, d\theta, \qquad n \in \mathbb{Z}.$$

Then, for nice enough functions f, we have

$$f(\theta) \approx \sum_{n=-N}^{N} \widehat{f}_n \, \varphi_n(\theta)$$

with the approximation getting better when N grows.

Note that the functions φ_n are orthogonal:

$$\langle \varphi_n, \varphi_n \rangle = \delta_{nm}.$$