**Applications of matrix computations**     Exercise 3, September 26–30, 2016
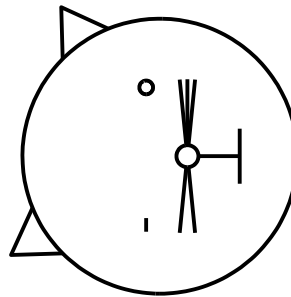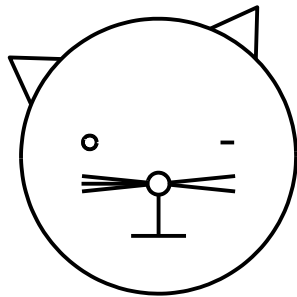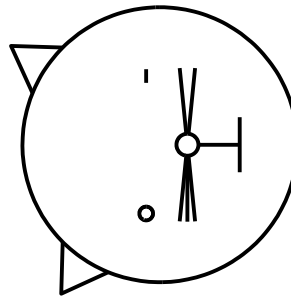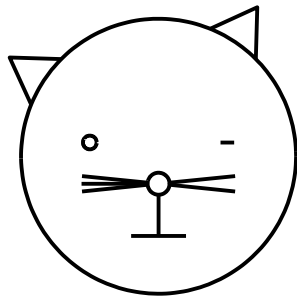
Lecturer: Samuli Siltanen.
Teaching assistants: Santeri Kaupinmäki and Jonatan Lehtonen.

Please send your solutions to application.matrixcomputation@gmail.com by Monday, October 3, at 10 AM.

1. Download the routine `kissa.m` from the course website. Find the two $2 \times 2$ matrices producing the following outputs with the `kissa.m`:

2. Define a matrix $A$ by

$$A = \begin{bmatrix} 5 & 2 \\ 2 & 5 \end{bmatrix}.$$

   (a) Determine the eigenvalues and eigenvectors of $A$ using the Matlab command `eig`.

   (b) Using the information from (a), write $A$ in the form $A = PDP^{-1}$ where $D$ is diagonal.

   (c) Demonstrate the action of the matrices $P$ and $D$ using the routine `kissa.m`. Explain what you see.

3. Consider the systems
$$Ax = b$$
of linear equations given below. Here $x = [x_1 \ x_2]^T$. Use Matlab to plot (in black color) the lines in the $(x_1, x_2)$ plane related to the equations. Also, plot the solution (or least-squares solution) as a red dot. Hint: try the commands `x = inv(A)*b` or `x = A\b` or `x = pinv(A)*b`.

(a) Take $A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$ and $b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

(b) Take $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Here a true solution does not exist (why?), so use least-squares solution.

(c) Take $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 1 \end{bmatrix}$ and $b = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$.

Here a true solution does not exist (why?), so use least-squares solution.

4. Download the file `DataFitExample1.m` from the course website. You will need to modify the file in the exercises below. Choose $a_0 = 0.8$ and $b_0 = 10$ in the ideal model $y = a_0 x + b_0$.

(a) Take the simulated measurement noise amplitude to be `noiselevel= 1;` and take the $x$ points to be `10:10:100;` leading to 10 data points. Compute the mean of ten least-squares estimated parameter values of $a$ and $b$. How close is the mean to the actual values? Repeat the above but take the mean over 100 recovered parameter values of $a$ and $b$. Is the mean now closer? (Hint: use `for`-loops.)

(b) Take the simulated measurement noise amplitude be `noiselevel= 5;` and take the $x$ points to be `10:1:100;` leading to 91 data points. Compute the mean of ten recovered parameter values of $a$ and $b$. How close is the mean to the actual values? Repeat the above but take the mean over 100 recovered parameter values of $a$ and $b$. Is the mean now closer? How do the results compare to those from (a)?

5. Consider the following ten-page internet, where arrows show links:
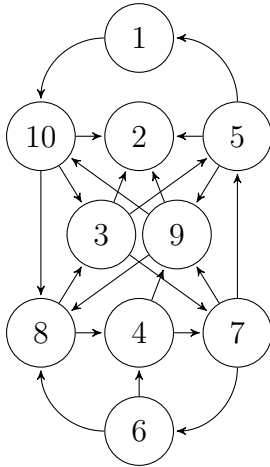


Use the PageRank algorithm as follows to find out the most important page.

(a) Construct the adjacency matrix $A = [a_{ij}]$, where $i$ is row index and $j$ is column index and

$$a_{ij} = \begin{cases} 1 & \text{if page } j \text{ has a link to page } i, \\ 0 & \text{otherwise.} \end{cases}$$

(b) Use the power method https://en.wikipedia.org/wiki/Power_iteration to find the dominant eigenvalue of $A$ and the corresponding eigenvector. Compute the eigenvalues and eigenvectors using the command `eig` and check that the result of the power method is approximately right.

(c) Normalize the eigenvector calculated in (b). Which web page is the most important?

6. Consider the Markov chain given by the teleoperator market share application discussed in the lecture. The explanation of the application is given on the course web page in the file *TeleOperatorComputation.pdf*.

We model the market situation by the horizontal vector $x = [x_1, x_2, x_3, x_4] \in \mathbb{R}^4$. Every component of $x$ satisfies $0 \leq x_j \leq 1$, and additionally

$$x_1 + x_2 + x_3 + x_4 = 1.$$

The interpretation is that $x_j$ is the market share of company $j$ at a time.

Given an initial state $x^{(0)}$, we get the market shares of the next day by

$$x^{(1)} = x^{(0)} P,$$

where $P = [p_{ij}]$ is the *transition matrix* with elements defined by

$$p_{ij} = \text{probability of a customer to change from company } i \text{ to company } j.$$

Now it is easy to see that on day $n$ the market shares are given by $x^{(n)} = x^{(0)} P^n$.

Take

$$P = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{3} & \frac{1}{6} \\ 0 & \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{10} & 0 & 0 & \frac{9}{10} \end{bmatrix}$$

Note that the matrix $P$ above differs from the one in the file *TeleOperator-Computation.pdf*. This is done on purpose.

(a) Take $x^{(0)} = [\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}]$ and compute $x^{(n)}$ for several values of $n$. How big needs $n$ to be for the ten first digits of $x^{(n-1)}$ and $x^{(n)}$ to agree? Hint: use the command `format long` before the computation.

(b) Choose a couple of other initial states $x^{(0)}$ and compute again $x^{(n)}$ for several values of $n$. How big needs $n$ to be for the ten first digits of $x^{(n-1)}$ and $x^{(n)}$ to agree?

(c) After the experiments (a) and (b) you should be convinced that there is a unique equilibrium state $x^{(\infty)} = \lim_{n \to \infty} x^{(n)}$. Find out how to write $x^{(\infty)}$ as an eigenvector of a matrix, and compute $x^{(\infty)}$ using Matlab's `eig` command.