Äärimmäisten ilmiöiden teoriaa, harj. 5, 28.11.-16

1. $\mathbb{E}(e^{sX_i}) = \int_0^\infty e^{sx}\mu_i e^{-\mu_i x}\,dx = \frac{\mu_i}{\mu_i - s}, \quad s < \mu_i$

$\Rightarrow \log \mathbb{E}(e^{sT_n}) = \sum_{i=1}^n \log \frac{\mu_i}{\mu_i - s},$

$\Lambda(s) = \log \frac{\mu}{\mu - s}, \quad s \in B(0,\varepsilon), \ \varepsilon > 0$ pieni

Siis $\Lambda'(s) = \frac{1}{\mu - s}$, $\Lambda'(0) = \frac{1}{\mu}$, väite seuraa lauseesta 3.4.

2. $\mathbb{P}(S_n/n > a) = (1+o(1))\, n\bar{F}(n(a-\mu)),$

$\mathbb{P}(S_n/n \geq b) \geq \mathbb{P}(S_n/n > b) = (1+o(1))\, n\bar{F}(n(b-\mu)),$

$\mathbb{P}(S_n/n \geq b) \leq \mathbb{P}(S_n/n > b-\varepsilon) = (1+o(1))\, n\bar{F}(n(b-\varepsilon-\mu))$

$= (1+o(1))\, n \left(\frac{b-\varepsilon-\mu}{b-\mu}\right)^{-\alpha} \bar{F}(n(b-\mu))$

Nähdään, että $\mathbb{P}(S_n/n \geq b) = (1+o(1))\, n\bar{F}(n(b-\mu))$ ja

$\mathbb{P}(S_n/n \in (a,b)) = \mathbb{P}(S_n/n > a) - \mathbb{P}(S_n/n \geq b)$

$= (1+o(1))\, n\bar{F}(n(a-\mu)) - (1+o(1))\, n\bar{F}(n(b-\mu))$

$= (1+o(1))\, n\bar{F}(n(a-\mu)) - (1+o(1)) \left(\frac{b-\mu}{a-\mu}\right)^{-\alpha} n\bar{F}(n(a-\mu))$

$= (1+o(1))\, n\bar{F}(n(a-\mu)) \left(1 - \left(\frac{a-\mu}{b-\mu}\right)^{\alpha}\right)$

3. $\mathbb{P}(S_n > n^x) \geq \mathbb{P}(M_n > n^x) = 1 - (1-\bar{F}(n^x))^n$

$= 1 - e^{-n\bar{F}(n^x) + o(1)\, n\bar{F}(n^x)} = (1+o(1))\, n\bar{F}(n^x)$

$\Rightarrow \liminf_{n\to\infty} (\log n)^{-1} \log \mathbb{P}(S_n > n^x)$

$\geq 1 + \frac{\log n^x}{\log n} \liminf_{n\to\infty} (\log n^x)^{-1} \log \bar{F}(n^x) = 1 - \alpha x.$

4. L. 3.11 $\Rightarrow$ $\mathbb{P}(M_n > nb, S_n > na) = (1+o(1))\,\mathbb{P}(S_n > na)$

$\Rightarrow \mathbb{P}(S_n + M_n > n(a+b) \mid S_n > na)$

$= \mathbb{P}(S_n > na)^{-1}\,\mathbb{P}(S_n + M_n > n(a+b), S_n > na)$

$\geq \mathbb{P}(S_n > na)^{-1}\,\mathbb{P}(S_n > na, M_n > nb) = 1 + o(1)$

5. L 3.3 $\Rightarrow$ $\mathbb{P}(S_n > n(a+\varepsilon)) = o(1)\,\mathbb{P}(S_n > na)$ ja

$\mathbb{P}(M_n > n\varepsilon, S_n > na) = o(1)\,\mathbb{P}(S_n > na)$

$\Rightarrow \mathbb{P}(S_n + M_n > n(a+b) \mid S_n > na)$

$= \mathbb{P}(S_n > na)^{-1}\,\mathbb{P}(S_n > na, S_n + M_n > n(a+b))$

$\leq \mathbb{P}(S_n > na)^{-1}\,\mathbb{P}(S_n > n(a+\varepsilon)$ tai $\{M_n > n\varepsilon, S_n > na\})$,

kun $\varepsilon < \frac{b}{2}$. Siis

$\mathbb{P}(S_n + M_n > n(a+b) \mid S_n > na)$

$\leq \mathbb{P}(S_n > na)^{-1}\,[o(1)\,\mathbb{P}(S_n > na) + o(1)\,\mathbb{P}(S_n > na)] = o(1)$.