Äärimmäisten ilmiöiden teoriaa, harj. 3 31.10.-16

1. $\bar{F}(x) = \int_x^\infty \frac{1}{\pi(1+y^2)}\, dy$,

$$\lim_{t\to\infty} \frac{\bar{F}(tx)}{\bar{F}(t)} \overset{H}{=} \lim_{t\to\infty} \frac{-x\cdot\frac{1}{1+t^2x^2}}{-\frac{1}{1+t^2}} = x^{-1}$$

Siis $\bar{F} \in R_{-1}$, II.

$\gamma_n = (\bar{F})^{-1}(\frac{1}{n})$ eli

$$\bar{F}(\gamma_n) = \int_{\gamma_n}^\infty \frac{1}{\pi(1+y^2)}\, dy = \frac{1}{n}$$

$\Rightarrow \frac{\pi}{n} = \frac{\pi}{2} - \arctan \gamma_n$ ja $\gamma_n = \tan\left(\frac{\pi}{2} - \frac{\pi}{n}\right)$,

$a_n = \gamma_n \sim \frac{1}{\cos(\frac{\pi}{2} - \frac{\pi}{n})} = \frac{1}{\sin\frac{\pi}{n}} \sim \frac{n}{\pi}$, $b_n = 0$

2. Jos $\xi, \xi_1, \xi_2, \ldots$ i.i.d. $N(0,1)$, $M'_n = \max(\xi_1, \ldots, \xi_n)$, niin

$$\mathbb{P}(M'_n \le a'_n x + b'_n) \xrightarrow{d} e^{-e^{-x}}.$$

Jos $X = e^{\xi}$, $X_i = e^{\xi_i}$, $M_n = \max(X_1, \ldots, X_n)$, niin

$$\mathbb{P}(M_n \le a_n x + b_n) = \mathbb{P}(M'_n \le \log(a_n x + b_n))$$

Nyt $a'_n = \dfrac{1}{\sqrt{2\log n}}$, $b'_n = \sqrt{2\log n} - \dfrac{\log\log n + \log(4\pi)}{2\sqrt{2\log n}}$.

Valitaan

$$a_n = a'_n e^{b'_n}, \quad b_n = e^{b'_n}$$

$$\Rightarrow \mathbb{P}(M_n \le a_n x + b_n) = \mathbb{P}(M'_n \le \log(e^{b'_n}(a'_n x + 1)))$$

$$= \mathbb{P}(M'_n \le \underbrace{\log(a'_n x + 1)}_{\sim a'_n x} + b'_n) \to e^{-e^{-x}} \Rightarrow \mathrm{I}.$$

3. $f(x) = \dfrac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} x^{a-1}(1-x)^{b-1}$, $0 < x < 1$, $a,b > 0$.

Mahdolliset I ja III, $x_F = 1$

III:

$$\lim_{t\to\infty} \frac{\int_{1-\frac{1}{tx}}^{1} y^{a-1}(1-y)^{b-1}\,dy}{\int_{1-\frac{1}{t}}^{1} y^{a-1}(1-y)^{b-1}\,dy}$$

$$\overset{H}{=} \lim_{t\to\infty} \frac{\frac{1}{xt^2}\left(1-\frac{1}{tx}\right)^{a-1}\left(\frac{1}{tx}\right)^{b-1}}{\frac{1}{t^2}\left(1-\frac{1}{t}\right)^{a-1}\left(\frac{1}{t}\right)^{b-1}} = x^{-b}, \quad \forall x > 0.$$

$\therefore$ III, $b_n = 1$,

$$\lim_{x\to 1-} \frac{\bar F(x)}{(1-x)^b} = \lim_{x\to 1-} \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \cdot \frac{x^{a-1}(1-x)^{b-1}}{b\cdot(1-x)^{b-1}}$$

$$= \frac{\Gamma(a+b)}{b\Gamma(a)\Gamma(b)}$$

Jos $\bar G(x) = \dfrac{(1-x)^b\,\Gamma(a+b)}{b\Gamma(a)\Gamma(b)}$, niin $\bar G(x) \sim \bar F(x)$, $x \to 1-$, ja

$\bar G(x) = \frac{1}{n} \Rightarrow (1-x)^b = \dfrac{b\Gamma(a)\Gamma(b)}{n\,\Gamma(a+b)}$

$\Rightarrow x = 1 - \dfrac{c}{n^{1/b}}$, $c = \left(\dfrac{b\Gamma(a)\Gamma(b)}{\Gamma(a+b)}\right)^{1/b}$

$\therefore a_n = \dfrac{c}{n^{1/b}}$, jos tarkastellaan G:tä. Ts.

$\lim_{n\to\infty} G^n(a_n x + b_n) = e^{-(-x)^b} \overset{L2.3}{\Rightarrow} \lim n\bar G(a_n x + b_n) = (-x)^b$, $x \le 0$

$\Rightarrow \lim n\bar F(a_n x + b_n) = (-x)^b \overset{L2.3}{\Rightarrow} \lim F^n(a_n x + b_n) = e^{-(-x)^b}$.

Eli siis yo. (a_n) käy normeerausjonoksi

4-5. $x_F = \infty$, joten I ja II mahdollisia. Nyt

$$\bar{F}(x) = (1-c)\sum_{k>x} c^k = c^{\lfloor x\rfloor+1}$$

I. $$\frac{\bar{F}(t+xg(t))}{\bar{F}(t)} = c^{\lfloor t+xg(t)\rfloor - \lfloor t\rfloor} = e^{(\lfloor t+xg(t)\rfloor - \lfloor t\rfloor)\log c}$$

$$\xrightarrow{t\to\infty} e^{-x} \iff \lfloor t+xg(t)\rfloor - \lfloor t\rfloor \to -\frac{x}{\log c}, \text{ kun } x \neq 0$$

Tämä ei ole mahdollista, jos $-\frac{x}{\log c} \notin \mathbb{N}$. ↯

II. $$\frac{\bar{F}(tx)}{\bar{F}(t)} = e^{(\lfloor tx\rfloor - \lfloor t\rfloor)\log c} \to 0, \text{ jos esim. } x=2 \text{ ↯}$$