Esim (Jatkoa edelliselle esimerkille (luvussa 7))

Ol $U, V \sim U(0,1)$, $U \perp\!\!\!\perp V$

Aset $\begin{cases} X = 3U \\ Y = \dfrac{U^2}{1-V} \end{cases}$

Tied. siis $f_{X,Y}(x,y) = 1\{0<x<3,\ y>\frac{1}{9}x^2\} \cdot \frac{1}{27} x^2 y^{-2}$

ja $f_X(x) = \frac{1}{3} 1\{0<x<3\}$

$f_Y(y) = \frac{1}{3}\left(1\{0<y\le 1\} \cdot y^{-1/2} + 1\{y>1\} \cdot y^{-2} \right)$

Laske

1) $f_{X|Y}$ (X:n ehdollinen tiheys, ehdolla $Y=y$)

2) $f_{Y|X}$ (Y:n —〃—, ehdolla $X=x$)

(*)→3) $\mathbb{E}(Y^\alpha \mid X=x)$, kun $0<x<3$, kun $\alpha = 1$

4) $\mathbb{E}(X^\alpha \mid Y=y)$, kun $y>0$, ja $\alpha = \frac{1}{2}$

Vast 1) $f_{X|Y}(x|y) = \dfrac{f_{X,Y}(x,y)}{f_Y(y)}$ kun $f_Y(y) > 0$

Nyt $f_Y(y) > 0 \iff \boxed{y>0}$ joten $\boxed{\text{ol } y>0}$

$$\Rightarrow f_{X|Y}(x|y) = \frac{1\{0<x<3,\ y>\frac{1}{9}x^2\} \frac{1}{27} x^2 y^{-2}}{\frac{1}{3}\left(1\{0<y\le 1\}\cdot y^{-1/2} + 1\{y>1\} y^{-2}\right)} \quad (\cdot 3)$$

$$= 3 \cdot 1\{0<x<3\} \cdot \frac{1}{9} x^2 \, 1\{y>\tfrac{1}{9}x^2\} \cdot y^{-2} \left(1\{0<y\le 1\} y^{1/2} + 1\{y>1\} y^2 \right)$$

$$= \frac{1}{9}x^2 \cdot 1\{0<x<3\} \left(1\{\tfrac{1}{9}x^2 < y \le 1\} y^{-3/2} + 1\{y>1\} y^{2-2} \right)$$

$$= \begin{cases} \frac{1}{9}x^2 y^{-3/2}, & 0 < x < 3 \text{ ja } \frac{1}{9}x^2 < y \le 1 \\ \frac{1}{9}x^2, & 0 < x < 3 \text{ ja } y > 1 \\ 0, & \text{muutoin (kun } y > 0) \end{cases}$$

2) $f_{Y|X}(y|x) = \dfrac{f_{X,Y}(x,y)}{f_X(x)}$ kun $f_X(x) > 0$

Nyt $f_X(x) > 0 \iff 0 < x < 3$ vaan OL $0 < x < 3$

toteutuu vamasti ol. nojalla

$$\Rightarrow f_{Y|X}(y|x) = \frac{1\{0 < x < 3, y > \frac{1}{9}x^2\} \frac{1}{27} x^2 y^{-2}}{\frac{1}{3} 1\{0 < x < 3\}}$$

$$= 1\{y > \tfrac{1}{9}x^2\} \cdot \tfrac{1}{27} \cdot 3 \cdot x^2 y^{-2}$$

$$= \begin{cases} \frac{1}{9}x^2 y^{-2}, & 0 < x < 3, \ y > \frac{1}{9}x^2 \\ 0, & \text{muulloin (kun } 0 < x < 3) \end{cases}$$

(kun $0 < x < 3$) on suluissa, sillä voimme

(kun $y > 0$) — " —

laajentaa ehdollisen tiheyden esim. 0:na myös niille x (ja y varl) joille ehd. tiheys ei ole suoraan määritelty (kts. luvun 8 kalvojen sivu 12 tai monisteen sivu 108 kaava (8.4))

3) OL että $f_{Y|X}$ on laajennettu (tai laajentamaton, ei merkitystä koska oletamme $0 < x < 3$)

ja OL $0 < x < 3$

$$E(Y^\alpha | X = x) = \int_{-\infty}^{\infty} y^\alpha f_{Y|X}(y|x)\,dy$$

(*) määr

$$= \frac{1}{9} \int_{-\infty}^{\infty} y^\alpha x^2 y^{-2} \, 1\{y > \tfrac{1}{9}x^2, 0 < x < 3\}\,dy$$

tot OL mukaan

$$= \frac{1}{9}x^2 \int_{\frac{1}{9}x^2}^{\infty} y^{-2+\alpha}\,dy$$

$\iff \{\frac{1}{9}x^2 < y < \infty\} \longrightarrow$ int. rajat

Jatkoa $\mathbb{E}(Y^\alpha \mid X = x) = \ldots = \frac{1}{9}x^2 \int_{1/9x^2}^{\infty} y^{-2+\alpha}\,dy$

$$= \frac{1}{9}x^2 \Big/_{1/9x^2}^{\infty} \left(\frac{y^{-1+\alpha}}{-1+\alpha} \mathbb{1}\{\alpha \neq 1\} + \ln y \cdot \mathbb{1}\{\alpha = 1\} \right)$$

$$= \begin{cases} \frac{1}{9}x^2 \Big/_{1/9x^2}^{\infty} \ln y = \infty \text{ (eli ei ehd. odotusarvo)} & \alpha = 1 \\ \frac{1}{9}x^2 \Big/_{1/9x^2}^{\infty} \frac{y^{-1/2}}{-1/2} = \frac{1}{9}x^2\left(-2\cdot 0 + 2\cdot \underbrace{\left(\tfrac{1}{9}x^2\right)^{-1/2}}_{=\sqrt{9x^{-2}} = 3x^{-1}}\right), & \alpha = \frac{1}{2} \end{cases}$$

$$= \begin{cases} \infty, & \alpha = 1 \\ \frac{1}{3}x, & \alpha = \frac{1}{2}. \end{cases}$$

4) Ol. $y > 0$

$$\mathbb{E}(X^\alpha \mid Y = y) \overset{\text{määr}}{=} \int_{-\infty}^{\infty} x^\alpha f_{X\mid Y}(x \mid y)\,dx$$

$$= \int_{-\infty}^{\infty} x^\alpha \cdot \Big(\tfrac{1}{9}x^2 y^{-3/2}\, \mathbb{1}\{0 < x < 3,\ \tfrac{1}{9}x^2 < y \le 1\} + \tfrac{1}{9}x^2 \cdot \mathbb{1}\{0 < x < 3,\ y > 1\} \Big)\,dx$$

(int. rajat)

Ol ensin $y > 1$

$$\Rightarrow \mathbb{E}(X^\alpha \mid Y = y) = \int_0^3 \frac{1}{9}x^2 x^\alpha \cdot \underbrace{\mathbb{1}\{y > 1\}}_{=1 \text{ ol. mukaan}}\,dx$$

$$= \Big/_0^3 \frac{1}{9}\frac{x^{3+\alpha}}{3+\alpha} = \frac{1}{9}\frac{3^{3+\alpha}}{3+\alpha}$$

$$= \begin{cases} \frac{1}{9}\cdot\frac{3^4}{4}, & \alpha = 1 \\ \frac{1}{9}\cdot\frac{3^3\sqrt{3}}{\underbrace{3+\frac{1}{2}}_{=\frac{7}{2}}}, & \alpha = \frac{1}{2} \end{cases} = \begin{cases} \frac{9}{4}, & \alpha = 1 \\ \frac{2}{9\cdot 7}\cdot 9\cdot 3\sqrt{3}, & \alpha = \frac{1}{2} \end{cases}$$

$$= \begin{cases} \frac{9}{4}, & \alpha = 1 \\ \frac{6\sqrt{3}}{7}, & \alpha = \frac{1}{2}. \end{cases}$$

Ol nyt $0 < y \leq 1$ $\Rightarrow$ $\mathbb{1}\{0 < x < 3, \frac{1}{9}x^2 < y \leq 1\}$

$\Rightarrow \frac{1}{9}x^2 < y$ $= \mathbb{1}\{0 < x < 3\sqrt{y}, 0 < y \leq 1\}$

$\Rightarrow x^2 < 9y$

$\Rightarrow x < 3\sqrt{y} < 1$

(oli päättelimme $\begin{cases} 0 < x < 3 \\ 1/9\, x^2 < y \leq 1 \end{cases} \Rightarrow \begin{cases} 0 < x < 3\sqrt{y} \\ 0 < y \leq 1 \end{cases}$

Toisaalta, jos $\begin{cases} 0 < x < 3\sqrt{y} \\ 0 < y \leq 1 \end{cases}$, niin $0 < x < 3\sqrt{y} \leq 3$ ja $\frac{1}{9}x^2 < y \leq 1$)

$\therefore$ kun $0 < y \leq 1$,

$$\mathbb{E}(X^\alpha \mid Y = y) = \int_{-\infty}^{\infty} x^\alpha \cdot \left(\frac{1}{9} x^2 y^{-3/2} \mathbb{1}\{0 < x < 3\sqrt{y}\} \cdot \mathbb{1}\{0 < y \leq 1\}\right) dx$$

↑ int. rajat; $\mathbb{1}\{0 < y \leq 1\} = 1$ ol. mukaan

$$= \frac{1}{9} y^{-3/2} \int_0^{3\sqrt{y}} x^{\alpha+2}\, dx$$

$$= \frac{1}{9} y^{-3/2} \Big/_0^{3\sqrt{y}} \frac{x^{3+\alpha}}{3+\alpha} = \begin{cases} \frac{1}{9} y^{-3/2} \frac{(3\sqrt{y})^4}{4} & \alpha = 1 \\ \frac{1}{9} y^{-3/2} \frac{(3\sqrt{y})^{7/2}}{7/2}, & \alpha = \frac{1}{2} \end{cases}$$

$$= \begin{cases} y^{-3/2} \cdot y^2 \cdot \frac{1}{4} \cdot 3^2, & \alpha = 1 \\ y^{-3/2} \cdot y^{7/4} \cdot 3^{-2} \cdot \frac{2}{7} \cdot 3^{7/2}, & \alpha = \frac{1}{2} \end{cases}$$

$$= \begin{cases} \frac{9}{4} y^{1/2}, & \alpha = 1 \\ \frac{6\sqrt{3}}{7} y^{1/4}, & \alpha = \frac{1}{2} \end{cases}$$

($= y^{\frac{1}{4}}$; $= 3^{3/2} = 3\sqrt{3}$)

$$\therefore \quad \mathbb{E}(X^\alpha \mid Y = y) = \begin{cases} \frac{9}{4} \mathbb{1}\{y > 1\} + \frac{9}{4} y^{1/2} \mathbb{1}\{0 < y \leq 1\}, & \alpha = 1 \\ \frac{6\sqrt{3}}{7} \mathbb{1}\{y > 1\} + \frac{6\sqrt{3}}{7} y^{1/4} \{0 < y \leq 1\}, & \alpha = \frac{1}{2} \end{cases}$$