Esim

Olk. $f(x,y) = c(\sin x + y^2)\,\mathbb{1}\{0 < x < \pi,\ -1 < y < 1\}$

0) onko f mahdollisesti yht. tf?

1) määrää c.

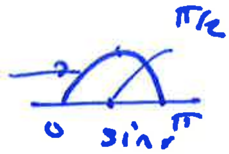
2) määrää f_X ja f_Y, kun $f_{X,Y} = f$.

Vast.

0) $f(x,y) = 0$ kun $(x,y) \notin \{0 < x < \pi,\ -1 < y < 1\}$

$y^2 \geq 0$ kun $(x,y) \in \{$ —— " —— $\}$

$\sin x \geq 0$ kun $(x,y) \in \{$ —— " —— $\}$



$\therefore\ c(\sin x + y^2)\,\mathbb{1}\{\dots\} \geq 0 \quad \forall x,y$ kunhan $c \geq 0$.

Edelleen

$$\iint_{\mathbb{R}^2} f(x,y)\,dx\,dy \overset{\text{lin.}}{=} c\iint_{\mathbb{R}^2} \sin x\,\mathbb{1}\{\dots\}\,dx\,dy + c\int_{\mathbb{R}^2} y^2\,\mathbb{1}\{..\}\,dx\,dy$$

Fubini

$$= c\int_{-\infty}^{\infty} dx\left(\sin x \int_{-\infty}^{\infty} \mathbb{1}\{\dots\}\,dy\right) + c\int_{-\infty}^{\infty} dy\left(y^2 \int_{-\infty}^{\infty} \mathbb{1}\{\dots\}\,dx\right) = c\int_0^{\pi} dx\,\sin x \int_{-1}^{1} dy + c\int_{-1}^{1} dy\,y^2 \int_0^{\pi} dx$$

$$= 2c\int_0^{\pi} \sin x\,dx + \pi c\int_{-1}^{1} y^2\,dy$$

parill. fn

$$= 2c \Big/_0^{\pi} -\cos x + 2\pi \cdot c \Big/_0^{1} y^3/3 = 2c(1-(-1)) + 2\pi c\left(\tfrac{1}{3} - 0\right)$$

$$= 4c + \frac{2\pi}{3}c \qquad \therefore\ 0 < \int_{\mathbb{R}^2} \dots < \infty \quad \text{kunhan } 0 < c < \infty.$$

$\therefore$ f on yht. tf. jollakin $c > 0$.

1) määrää c.

koska f on ~~tth~~ ytf, niin $1 = \iint_{\mathbb{R}^2} f(x,y)\,dx\,dy$

$\underset{\uparrow\,(0)}{=} 4c + \frac{2\pi}{3}c = \left(4 + \frac{2\pi}{3}\right)c = \left(\frac{12+2\pi}{3}\right)c$

$\therefore\ c = \frac{3}{12+2\pi}.$

2) $f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy = \frac{3}{12+2\pi}\int_{-\infty}^{\infty}(\sin x + y^2)\,1\{\dots\}\,dy$

lin.

$= \frac{3}{12+2\pi}\int_{-\infty}^{\infty}\sin x\,1\{\dots\}\,dy + \frac{3}{12+2\pi}\int_{-\infty}^{\infty} y^2\,1\{\dots\}\,dy$

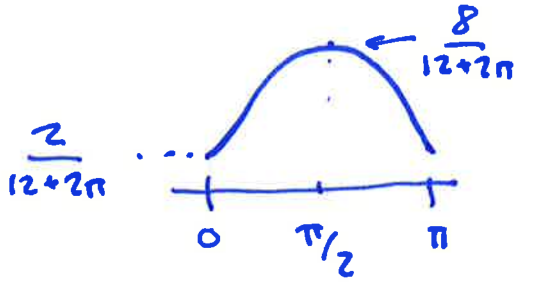
Nyt $1\{\dots\} = 1\{0<x<\pi,\ -1<y<1\}$

$= 1\{0<x<\pi\}\cdot 1\{-1<y<1\}$

$\therefore\ f_X(x) = \frac{3}{12+2\pi}\Bigg(\int_{-1}^{1}\sin x\ 1\{0<x<\pi\}\,dy$

$+ \underbrace{\int_{-1}^{1} y^2\ 1\{0<x<\pi\}\,dy}_{=1\{\dots\}\cdot 2\cdot\int_0^1 y^2\,dy}\Bigg) = \frac{6\sin x\ 1\{0<x<\pi\}}{12+2\pi}$

$+ \frac{\cancel{3}\cdot\frac{1}{\cancel{3}}\cdot 2}{\cancel{3}\,\cancel{3}\cdot(12+2\pi)}\,1\{0<x<\pi\} = 1\{0<x<\pi\}\cdot\left(\frac{6\sin x + 2}{12+2\pi}\right)$



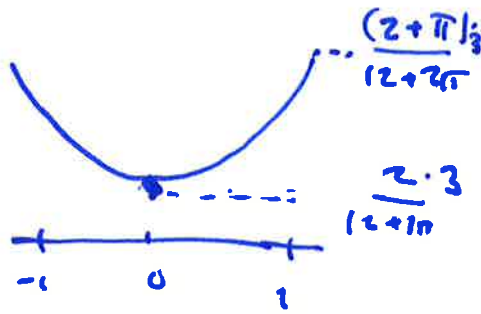
$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx = \frac{3}{12+2\pi} \int_{-\infty}^{\infty} (\sin x + y^2)\, 1\!\!1\{\ldots\}\,dx$$

lin

$$= \ldots \underset{1\!\!1\{\ldots\} = 1\!\!1\{\ldots\}\cdot 1\!\!1\{\ldots\}}{=} \ldots = \frac{3}{12+2\pi} \int_0^{\pi} (\sin x + y^2)\, 1\!\!1\{-1<y<1\}\,dx$$

$$= \frac{3}{12+2\pi}\, 1\!\!1\{-1<y<1\} \left(\Big/_{0}^{\pi} -\cos x + xy^2 \right)$$

$$= \quad - \quad || \quad - \quad (2 + \pi y^2)$$



Esim

Olk. (X, Y) kuten edellisessä esimerkissä

(eli $f_{X,Y} = f$)

1) Määrää odotusarvovektori

Vast

$$\mathbb{E}X = \int_{-\infty}^{\infty} x f_X(x)\,dx \underset{\substack{\llcorner 3)\\ \text{edellä}}}{=} \int_0^{\pi} \frac{(6\sin x + 2)x}{12+2\pi}\,dx$$

Nyt ositt. integrointi $\int_0^{\pi} \sin x \cdot x\,dx = \int_0^{\pi} x \frac{\partial}{\partial x}(-\cos x)\,dx$

$$= \Big/_{0}^{\pi} x\cdot(-\cos x) - \int_0^{\pi} (-\cos x)\cdot \underbrace{\frac{\partial}{\partial x} x}_{=1}\,dx$$

$$= \pi\cdot(+1) - 0 + \int_0^{\pi} \cos x\,dx = \pi + \Big/_{0}^{\pi} \sin x = \pi + \underbrace{(0-0)}_{=0}$$

Joten $\int_0^{\pi} (6\sin x + 2)x\,dx = 6\pi + \int_0^{\pi} 2x\,dx = 6\pi + \pi^2$

$$\therefore \mathbb{E}X = \frac{6\pi + \pi^2}{12+2\pi} = \frac{\pi}{2}\left(\frac{6+\pi}{6+\pi}\right) = \frac{\pi}{2}$$

$$\mathbb{E}\,Y \underset{\substack{\uparrow \\ 31 \\ \text{edellä}}}{=} \ldots \int_{\cancel{-\infty}\,-1}^{\cancel{\infty}\,1} \frac{3}{12+2\pi}\cdot(2+\pi y^2)\cdot y\,dy$$

$$= \int_{-1}^{1} \underbrace{\frac{6}{12+2\pi}\,y}_{\text{pariton}} + \underbrace{\frac{6\pi}{12+2\pi}\,y^3}_{\text{pariton}} = 0.$$

$$\therefore\quad \mathbb{E}\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} \frac{\pi}{2} \\ 0 \end{pmatrix}$$

2) Määrää $\mathrm{Cov}\left(\begin{pmatrix} X \\ Y \end{pmatrix}\right)$ eli satunnaisvektorin kovarianssimatriisi

Vast. Koska luentojen mukaan

$$\mathrm{Cov}\left(\begin{pmatrix} X \\ Y \end{pmatrix}\right) = \begin{pmatrix} \mathrm{Var}\,X & \mathrm{cov}(X,Y) \\ \mathrm{cov}(X,Y) & \mathrm{Var}\,Y \end{pmatrix}$$

niin tarvitsemme:

- $\mathrm{Var}\,X = \mathbb{E}X^2 - (\mathbb{E}X)^2$ ← tarvitsemme tämän
- $\mathrm{Var}\,Y = \mathbb{E}Y^2 - \underbrace{(\mathbb{E}Y)^2}_{=0}$ ← tarvitsemme tämän
- $\mathrm{cov}(X,Y) = \mathbb{E}(XY) - \mathbb{E}X\,\mathbb{E}Y$ ← tarvitsemme tämän.

$$\mathbb{E}\,Y^2 = \int_{-1}^{1} \frac{3}{12+2\pi}(2+\pi y^2)y^2\,dy$$

$$\underset{\substack{\uparrow \\ y^2,\,y^4 \\ \text{parillisia}}}{=} 2\int_0^1 \frac{3}{12+2\pi}(2y^2+\pi y^4)\,dy = \frac{6}{12+2\pi}\Big/_0^1 \frac{2y^3}{3} + \frac{\pi y^5}{5}$$

$$= \frac{4+\frac{6}{5}\pi}{12+2\pi} = \frac{20+6\pi}{5(12+2\pi)}$$

$$= \frac{10+3\pi}{5\cdot(6+\pi)}$$

$$\mathbb{E}X^2 = \int_0^{\pi} \frac{3(\sin x)x^2 + x^3}{6+\pi}\,dx$$

$$= \frac{\pi^4}{4\cdot(6+\pi)} + \frac{3}{6+\pi}\underbrace{\int_0^{\pi} x^2 \sin x\,dx}$$

$$\overset{\text{osiH.ind}}{=} \Big/_0^{\pi}\, x^2(-\cos x) + \int_0^{\pi} 2x\cos x\,dx$$

$$\overset{\text{osiH.ind.}}{=} \pi^2 + \underbrace{\Big/_0^{\pi}\, 2x\sin x}_{=0} - \int_0^{\pi} 2\sin x\,dx$$

$$= \pi^2 - 2\Big/_0^{\pi}(-\cos x) = \pi^2 - 4$$

$\therefore$ $$\mathbb{E}X^2 = \left(\frac{\pi^4}{4} + 3(\pi^2-4)\right)\cdot\frac{1}{6+\pi}$$

$$= \frac{\pi^4 + 12\pi^2 - 16}{4}\cdot\frac{1}{6+\pi}$$

$\therefore$ $$\operatorname{Var} Y = \frac{10+3\pi}{5(6+\pi)}$$

$$\operatorname{Var} X = \frac{1}{6+\pi}\left(\frac{\pi^4+12\pi^2-16}{4}\right) - \frac{\pi^2}{4}$$

$$= \frac{\pi^4 - 6\pi^2 - \pi^3 + 12\pi^2 - 16}{4(6+\pi)}$$

$$= \frac{\pi^4 + 6\pi^2 - \pi^3 - 16}{24+4\pi} = \frac{\pi^3\overbrace{(\pi-1)}^{>0} + 2\overbrace{(3\pi^2-8)}^{>0}}{24+4\pi}$$

$$\mathbb{E}(XY) = \iint_{\mathbb{R}^2} f_{X,Y}(x,y)\,xy\,dx\,dy = \int_{-1}^{1} dy \int_0^{\pi} dx\, \frac{3}{12+2\pi}\left(\sin x\cdot xy + xy^3\right)$$

Pa

paridon

$$= 0.$$

$\therefore$ $$\operatorname{cov}(X,Y) = \mathbb{E}(XY) = 0$$

$\therefore$ $$\operatorname{Cov}\left(\begin{pmatrix}X\\Y\end{pmatrix}\right) = \begin{pmatrix} \frac{\pi^3(\pi-1)+2(3\pi^2-8)}{24+4\pi} & 0 \\ 0 & \frac{10+3\pi}{30+5\pi} \end{pmatrix}$$