Standardi normaalijakauma $N_n(0, I)$

Määr. $U = (U_1, \dots, U_n)$ $\quad U_i \sim N(0,1)$ $\forall i$

ja $\{U_1, \dots, U_n\}$ ⫫

Ominaisuuksia

- $\mathbb{E}U = \begin{pmatrix} \mathbb{E}U_1 \\ \vdots \\ \mathbb{E}U_n \end{pmatrix} = \bar{0} = 0_n$ laitan vekt. päälle viivan tai lihavoin kaulalla

- $\operatorname{Cov} U \overset{\text{määr.}}{=} \left(\operatorname{cov}(U_i, U_j) \right)_{i,j=1}^{n} \overset{⫫}{=} \left(1\{i=j\} \right)_{i,j=1}^{n} = I_n.$

- $M_U(\bar{t}) = \mathbb{E}\exp(\bar{t}^T U) \overset{\text{matr. tulon määr.}}{=} \mathbb{E}\exp\left(\sum_i t_i U_i\right)$

$$= \mathbb{E}\prod_{i=1}^{n} \exp(t_i U_i) \overset{⫫}{=} \prod_{i=1}^{n} \mathbb{E}\exp(t_i U_i)$$

$$= \prod_{i=1}^{n} \exp\left(\tfrac{1}{2} t_i^2\right) = \exp\left(\tfrac{1}{2}\sum_{i=1}^{n} t_i^2\right)$$

$U_i \sim N(0,1)$

$$= \exp\left(\tfrac{1}{2}\bar{t}^T \bar{t}\right).$$

Yleinen multinorm. jakauma

Määr. $X \sim N_m(\underbrace{\mu}_{\mathbb{E}X}, \underbrace{\Sigma}_{\operatorname{Cov}(X)}) \iff \exists$ stand. norm. jak. $U \sim N_n(0, I_n)$

$\exists A$, $m \times n$-matriisi ja $\exists \mu \in \mathbb{R}^m$ s.e.

$X \overset{d}{=} AU + \mu$ $\qquad$ (ja $\Sigma = AA^T$)

Om. $\mathbb{E}X = A\underbrace{\mathbb{E}U}_{=0} + \mu = \mu$

$\operatorname{Cov}(X) = \operatorname{Cov}(AU + \mu) = \operatorname{Cov}(AU) = A \operatorname{Cov} U A^T$

$$= AA^T = \Sigma.$$

$$M_{\underline{X}}(\bar{t}) = \mathbb{E}\exp(\bar{t}^T\underline{X}) = \mathbb{E}\exp(\bar{t}^T(AU+\bar{\mu}))$$

$$= \mathbb{E}\exp(\bar{t}^T AU + \bar{t}^T\bar{\mu}) = \mathbb{E}\exp(\bar{t}^T AU) \cdot \exp(\bar{t}^T\bar{\mu})$$

$$= \mathbb{E}\exp(\underbrace{(A^T\bar{t})^T}_{=t^T A^{TT}} U) \cdot \exp(\bar{t}^T\bar{\mu})$$

$$= M_U(A^T\bar{t})\exp(\bar{t}^T\bar{\mu}) = \exp(\tfrac{1}{2}(A^T\bar{t})^T(A^T\bar{t})) \cdot \exp(\bar{t}^T\bar{\mu})$$

$$= \exp(\tfrac{1}{2}\bar{t}^T \underbrace{AA^T}_{=\Sigma}\bar{t} + \bar{t}^T\bar{\mu})$$

<u>Lause</u> $\underline{X} \sim N_m \Rightarrow$ i) $\mathbb{E}\underline{X} = \mu$

$\underline{X} = AU + \bar{\mu}$ ii) $\mathrm{Cov}\,\underline{X} = AA^T = \Sigma$

$U \sim N_n(0, I_n)$ iii) jakauma määr. yks. od.-arvovekt. ja kovarianssimat.

iv) ~~dos~~ os $\mu \in \mathbb{R}^m$

$\Sigma \in \mathbb{R}^{m\times m}$ pos. semid. matriisi

$\Rightarrow \exists\, \underline{X}$ s.v. jolla on multin. jakauma $\mathbb{E}\underline{X} = \mu$, $\mathrm{Cov}(\underline{X}) = \Sigma$.

<u>Tod</u> i), ii) on

iii) jos $\underline{X} \sim N_m$ $\Rightarrow$ $M_{\underline{X}} = M_{\underline{Y}}$ määräytyy $\Rightarrow$ $\underline{X}$:llä ja $\underline{Y}$:llä sama jakauma.

$\underline{Y} \sim N_m$

ja $\mathbb{E}\underline{X} = \mathbb{E}\underline{Y}$

$\mathrm{Cov}\,\underline{X} = \mathrm{cov}\,\underline{Y}$

iv) Jos $\mu \in \mathbb{R}^m$ ja $\Sigma \in \mathbb{R}^{m\times m}$ ~~matr~~ ja pos semidef.

Cholesky

$\Rightarrow \exists A$ st $\Sigma = AA^T$. $\Rightarrow \underline{X} = AU + \mu$ käy.

⇒

Simulointi $N_n(\mu, \Sigma)$

- Σ sym. pos. semidef.
- etsi A s.e. $\Sigma = AA^T$ (esim. Cholesky)
- simuloidaan n riipp. arvoa $(u_1, \dots, u_n)$ $N(0,1)$:stä (∴ saadaan sim. $\bar U$:lle) yksi arvo $\bar u = (u_1, \dots, u_n)$
- Lasketaan $\bar x = A\bar u + \bar\mu$.

Lause Affiininen muunnos

$X \sim N_n(\bar\mu, \Sigma)$
$B \in \mathbb{R}^{p \times n}$ vakiomatr.
$\bar b \in \mathbb{R}^p$

$\Rightarrow \quad BX + b \sim N_p(B\bar\mu + b, B\Sigma B^T)$

Tod

$$BX + \bar b \overset{d}{=} B(A\bar U + \bar\mu) + \bar b = (BA)\bar U + (B\bar\mu + \bar b)$$

$\therefore \quad BX + \bar b \sim N_p$

koska
- $\mathbb{E}(BX + \bar b) = B\,\mathbb{E}X + \bar b = B\bar\mu + \bar b$
- $\mathrm{Cov}(\quad) = B\,\mathrm{Cov}\,X\,B^T = B\Sigma B^T \Rightarrow \square$

Reunajakauma on multinormaalijak.

$$X = (X_1, \dots, X_n) = (X_i)_{i=1}^n$$

Koska $X_i = \bar e_i^T X$ kun $\bar e_i = (0, 0, \dots, 1, 0, \dots, 0)$ ↑ i:s komponentti

$\Rightarrow$ jos $Y = (X_{i_1}, \dots, X_{i_k})$

niin

$$Y = \begin{pmatrix} X_{i_1} \\ \vdots \\ X_{i_k} \end{pmatrix} = \begin{pmatrix} \bar e_{i_1}^T X \\ \vdots \\ \bar e_{i_k}^T X \end{pmatrix} \overset{\text{mat. tulo}}{=} \begin{pmatrix} \bar e_{i_1}^T \\ \vdots \\ \bar e_{i_k}^T \end{pmatrix} X$$

$X = (X_1, \dots, X_4)$
$Y = (X_2, X_3)$

$$= \begin{pmatrix} X_2 \\ X_3 \end{pmatrix} = \begin{pmatrix} \bar e_2^T \\ \bar e_3^T \end{pmatrix} X = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \cdot X$$

$\Rightarrow$ reunajakauma (eli Y:n jakauma) on lin. muun. $\Rightarrow$ on multi.

Edellen

$$\mathbb{E}\,\bar{Y} = \begin{pmatrix} \bar{e}_{i_1}^T \\ \vdots \\ \bar{e}_{i_n}^T \end{pmatrix} \mathbb{E}\bar{X} = \begin{pmatrix} \bar{e}_{i_1}^T \bar{\mu} \\ \vdots \\ \bar{e}_{i_n}^T \bar{\mu} \end{pmatrix} \overset{\text{Huom}}{=} \begin{pmatrix} \mu_{i_1} \\ \vdots \\ \mu_{i_n} \end{pmatrix}$$

$\operatorname{Cov}(\bar{Y}) =$ muttu helpommin suoraan

$$\big(\underbrace{\operatorname{cov}(X_{i_p}, X_{i_q})}_{= \Sigma_{i_p, i_q}}\big)_{p,q=1}^{n}$$

$$\mathbb{E}\bar{Y} = \mathbb{E}\begin{pmatrix} X_{i_1} \\ \vdots \\ X_{i_n} \end{pmatrix}$$

Esim. $\mathbb{E}\,\bar{Y} = \begin{pmatrix} \mathbb{E}X_2 \\ \mathbb{E}X_3 \end{pmatrix}$ ja $\operatorname{Cov}\bar{Y} = \begin{pmatrix} \operatorname{Var} X_2 & \operatorname{cov}(X_2, X_3) \\ \operatorname{cov}(X_3, X_2) & \operatorname{Var} X_3 \end{pmatrix}$

Tiheysfunktio Σ pos. definiitti ja symm.

$$f(\bar{x}) = (2\pi)^{-m/2} (\det \Sigma)^{-1/2} \exp\left(-\tfrac{1}{2}(\bar{x}-\bar{\mu})^T \Sigma^{-1} (\bar{x}-\bar{\mu})\right)$$

Tod. Kun $\Sigma = AA^T$ ja A^{-1} on olem.

mii. $\bar{X} = A\bar{U} + \mu$. $\bar{U} = A^{-1}\bar{X} - A^{-1}\mu$ $\quad (\bar{U} = h(\bar{X}))$

"Laskuna" viimeksi että diff. kuvauksen

$h(\bar{y}) = A^{-1}(\bar{y} - \bar{b})$ der. matriisi on A^{-1}

(muistin der. matriisi on

$$\begin{pmatrix} D_1 h_1(\bar{y}) & \dots & D_n h_1(\bar{y}) \\ \vdots & & \vdots \\ D_1 h_n(\bar{y}) & \dots & D_n h_n(\bar{y}) \end{pmatrix} = \big(D_j h_i(\bar{y})\big)_{\underset{\uparrow\,\text{rivi}}{i},\, \underset{\text{sarake}}{j}=1}^{n}$$

$$= \Big(D_j \underbrace{\big(\textstyle\sum_{k=1}^{n} A^{-1}_{ik} (\bar{y} - \bar{b})\big)}_{= A^{-1}(\bar{x}-\mu)}\Big)_{i,j=1}^{n} = \underbrace{\big(A^{-1}_{ij}\big)_{i,j=1}^{n}}_{= A^{-1}}\quad)$$

$\Rightarrow$ Lause

$$f_{\bar{X}}(\bar{x}) = f_{\bar{U}}(\widetilde{h(\bar{x})})\,|\det A^{-1}|$$

$$= (2\pi)^{-m/2} \exp\Big(-\tfrac{1}{2}\underbrace{(A^{-1}(\bar{x}-\bar{\mu}))^T}_{= (\bar{x}-\bar{\mu})^T (A^{-1})^T} (A^{-1}(\bar{x}-\bar{\mu}))\Big)$$

$$\cdot \underbrace{|\det A|^{-1}}_{= \sqrt{\det \Sigma}}$$

sillä $\det(AA^T) = \det A \cdot \det A$

Korreloimattomuus ja riippumattomuus

Aina ~~riippumatt~~

$X \perp\!\!\!\perp Y \implies$ ~~ei korreloi~~ Y ja X ei korreloi

$\impliedby$ ei yleensä

mutta jos (X,Y) normaalijak. niin kyllä

Lause

Jos (X,Y) norm. jak. ja $\operatorname{cov}(X,Y)=0$ $\iff X \perp\!\!\!\perp Y$.

$$Z = \begin{pmatrix} X \\ Y \end{pmatrix} \quad \bar\mu = \mathbb{E}Z = \begin{pmatrix} \bar\mu_X \\ \bar\mu_Y \end{pmatrix} = \begin{pmatrix} \mathbb{E}X \\ \mathbb{E}Y \end{pmatrix}$$

$$= (X,Y) \quad \operatorname{Cov} Z = \Sigma$$

$$\Sigma = \begin{pmatrix} \Sigma_{XX} & \Sigma_{XY} \\ \Sigma_{YX} & \Sigma_{YY} \end{pmatrix}$$

Tod "$\impliedby$" on aina voimassa.

$\implies$

$$\bar t = (\bar u, \bar v)$$

$$M_Z(\bar t) = M_{X,Y}(\bar u, \bar v)$$

$$= \mathbb{E} \exp\left((\bar u \ \bar v) \begin{pmatrix} X \\ Y \end{pmatrix} \right)$$

$$= \mathbb{E} \exp(\bar u X + \bar v Y) = \mathbb{E}(\exp \bar u X)(\exp(\bar v Y))$$

$$\left(\overset{\text{jos } \perp\!\!\!\perp}{=} \mathbb{E}(\quad)\mathbb{E}(\quad) = M_X(\bar u) M_Y(\bar v) \right)$$

summan Z:n m.

$$= \exp\left(\tfrac{1}{2} \bar t^T \Sigma_Z \bar t + \bar t^T \bar\mu \right)$$

$$= \exp\left(\tfrac{1}{2} (\bar u^T \ \bar v^T) \begin{pmatrix} \Sigma_{XX} & \overset{0}{\Sigma_{XY}} \\ \underset{0}{\Sigma_{YX}} & \Sigma_{YY} \end{pmatrix} \begin{pmatrix} \bar u \\ \bar v \end{pmatrix} + (\bar u^T \ \bar v^T) \begin{pmatrix} \bar\mu_X \\ \bar\mu_Y \end{pmatrix} \right)$$

$$\overset{\text{ol.}}{=} \exp\left(\tfrac{1}{2} (\bar u^T \ \bar v^T) \begin{pmatrix} \Sigma_{XX} \bar u \\ \Sigma_{YY} \bar v \end{pmatrix} + (\bar u^T \bar v^T) \begin{pmatrix} \bar\mu_X \\ \bar\mu_Y \end{pmatrix} \right)$$

$$= \ldots$$

$$= \exp\left(\tfrac{1}{2} \bar u^T \Sigma_{XX} \bar u \right) \exp\left(\tfrac{1}{2} \bar v^T \Sigma_{YY} \bar v \right) \cdot \exp(\bar u^T \bar\mu_X) \cdot \exp(\bar v^T \bar\mu_Y)$$

$$= M_X(\bar u) M_Y(\bar v) \implies \square$$

Laus samaan ed. sillä jos

$X \perp\!\!\!\perp Y$, $X \sim N(\mu_X, \Sigma_{XX})$
$Y \sim N(\mu_Y, \Sigma_{YY})$

niin

$$M_X(\bar u) M_Y(\bar v) \overset{\perp\!\!\!\perp}{=} M_{(X,Y)}(\bar u, v)$$

$$\exp(\dots)\exp(\dots) = \exp(\dots + \dots)$$

Laus $(X,Y) \sim N(\mu, \Sigma)$, Σ_{XX} sääm.

$$Y \mid (X = x) \sim N\left(\bar\mu_Y + \Sigma_{YX}\Sigma_{XX}^{-1}(\bar x - \mu_X), \; \Sigma_{YY} - \Sigma_{YX}\Sigma_{XX}^{-1}\Sigma_{XY}\right)$$

Tod.

Ajatus on yksinkertainen (käyt. riippumattomuutta!?)
??

1) Merk. $V = Y - BX$. | X m-ul. Y n-ul. $B \in \mathbb{R}^{n \times m}$

2) Et. sell. B että $\mathrm{cov}(V, X) = 0$

3) V saadaan aff. muunnoksena (X,Y):stä

$\Rightarrow$ V multinor. jak. $V = \underbrace{(-B \,\vdots\, I)}_{=A}\begin{pmatrix} X \\ Y \end{pmatrix}$ | A n riviä ja m+n saraketta

(mutta tämä ei riitä!!,

Myös (V, X) saadaan aff. m. (X,Y):stä

$\Rightarrow (V,X)$ multinorm. jakauma |

$$\begin{pmatrix} V \\ X \end{pmatrix} = \begin{pmatrix} V \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ X \end{pmatrix} = \begin{array}{c} \\ n \\ m \end{array}\!\!\begin{array}{c} \begin{array}{cc} m & n \end{array} \\ \left(\begin{array}{c:c} -B & I \\ \hdashline 0 & 0 \end{array}\right) \end{array}\begin{pmatrix} X \\ Y \end{pmatrix} + \left(\begin{array}{c:c} 0_{n\times m} & 0_{n\times n} \\ \hdashline I_m & 0_{m\times n} \end{array}\right)\begin{pmatrix} X \\ Y \end{pmatrix}$$

$$= \hat A \begin{pmatrix} X \\ Y \end{pmatrix}$$

missä

$$\hat A = \left(\begin{array}{c:c} -B & I_n \\ \hdashline I_m & 0_{m\times n} \end{array}\right)$$

4) $2) + 3) \Rightarrow V \perp\!\!\!\perp X$.

$\underline{\text{lause}}$

korr. $\Rightarrow$

$\perp\!\!\!\perp$

5) $V = Y - BX, \quad V \perp\!\!\!\perp X$

$\Rightarrow Y = V + BX, \quad V \perp\!\!\!\perp X$

$\Rightarrow Y \mid (X = \bar{x}) \sim V + \underbrace{B\bar{x}}_{\text{vakiovektori}} \quad \Rightarrow Y \mid (X = \bar{x}) \sim N_n$

multinorm

n-ulott.

6) $\mathbb{E}(Y \mid X = \bar{x}) \underset{5)}{=} \mathbb{E}(V + B\bar{x}) = B\bar{x} + \underline{\mathbb{E}V}$

$\mathrm{Cov}(Y \mid X = \bar{x}) \underset{5)}{=} \mathrm{Cov}(V + B\bar{x}) = \underbrace{\mathrm{Cov}\, V}_{\text{funk.}}$

7) $\mathbb{E}V = \overset{\text{ulkoa}}{\mathbb{E}(Y - BX)} = \mathbb{E}Y - B\,\mathbb{E}X = \mu_Y - B\mu_X$

$\mathrm{Cov}(V) = \mathrm{cov}(Y - BX, Y - BX)$

$= \underbrace{\mathrm{Cov}\, Y}_{= \Sigma_{YY}} - \underbrace{\mathrm{cov}(Y, BX)}_{= \mathbb{E}(Y - \mathbb{E}Y)\cdot(BX - B\,\mathbb{E}X)^T = (X - \mathbb{E}X)^T B^T} - \mathrm{cov}(BX, Y) + \underbrace{\mathrm{Cov}(BX)}_{= B\,\mathrm{Cov}\,X\,B^T}$

lause 9 omin.

$= \Sigma_{YY}$

$- \Sigma_{YX} B^T - B\Sigma_{XY} + B\Sigma_{XX}B^T$.

eli kun han kohta 2) on tehty on homma valmis

2) $\mathrm{cov}(V, X) = \mathbb{E}(V - \mathbb{E}V)(X - \mathbb{E}X)^T$

lause 9 om.

$\mathrm{Cov}(Y - BX, X) = \mathrm{cov}(Y, X) - B\,\mathrm{cov}(X, X)$

$= \Sigma_{YX} - B\Sigma_{XX} = 0 \iff B\Sigma_{XX} = \Sigma_{YX}$ jos olem

$\iff B = \Sigma_{YX}\Sigma_{XX}^{-1}$