Statistical methods in public health
Adjusted means based on regression models and delta method

Tommi Härkänen

National Institute for Health and Welfare (THL)
Department of Health (TERO)

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Revisit back-door adjustment

Back-door adjustment

If a set of variables Z satisfies the back-door criterion relative to (X, Y), then the causal effect of X on Y is identifiable and is given by the formula

$$\mathbb{P}\{y \mid x\} = \sum_z \mathbb{P}\{y \mid x, z\} \mathbb{P}\{z\}. \tag{1}$$

Note that in (1)

Intervention $\mathbb{P}\{y \mid x\}$ is the predicted probability of $Y = y$ when the value of X is fixed to x. (Pearl uses notation $do(x)$)

(Direct) standardization The r.h.s. is a weighted average of probabilities $\mathbb{P}\{y \mid x, z\}$ estimated from subsets (x, z) and $\mathbb{P}\{z\}$ prevalence of the blocking variables $Z = z$.

Approximating a function by Taylor series

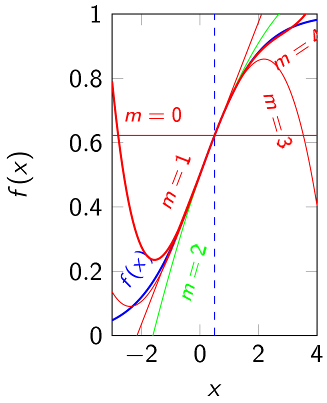
Taylor series

The Taylor series of a real or complex-valued function $f(x)$ that is infinitely differentiable at a real or complex number a is the power series

$$f(x) \approx \sum_{k=0}^{m} \frac{f^{(k)}(a)}{k!}(x-a)^k. \tag{2}$$

(Derivative of order zero $f^{(0)} := f$, $0! := 1$, $(x-a)^0 := 1$.)

For example, let $f(x) := 1/(1 + \exp\{-x\})$ and $a := 0.5$:



Delta method

Recall that if X is a random variable with $\mathbb{E}[X] = \mu$ and $\mathrm{Var}(X) = \sigma^2$. Then

- $\mathbb{E}[c(X - a)] = c(\mu - a)$ and
- $\mathrm{Var}\big(c(X - a)\big) = c^2\sigma^2$.

Recall the first terms of the Taylor series:

$$f(X) \approx f(a) + f^{(1)}(a)(X - a) + o(X - a)$$

Choose $a := \mu$. Then expectation of 2^{nd} term equals zero, and variance $\mathrm{Var}(X)[f^{(1)}(\mu)]^2$. Remainder $o(X - \mu)$ can be omitted if X is near a.

Univariate delta method

If for a sequence of random variables X_n $\sqrt{n}(X_n - \mu) \to N(0, \sigma^2)$, then

$$\sqrt{n}\big(f(X_n) - f(\mu)\big) \to N\big(0, \sigma^2[f^{(1)}(\mu)]^2\big). \tag{3}$$

Note that (3) can be generalized to random vectors B_n with mean vector β, covariance matrix Σ, and gradient vector ∇f:

$$\sqrt{n}\big(f(B_n) - f(\beta)\big) \to N\big(0, \nabla f(\beta)^T \Sigma \nabla f(\beta)\big).$$

Population Attributable Fraction (PAF)

Background

The importance of a risk factor in public health can be phrased e.g. *"How many disease cases could be avoided if risk factor X had been removed from a population?"*

- Two important aspects:

  Individual effect: How strong is the association of X with the disease?

  Prevalence: How common X is in the population?

  Risk factor with low individual effect but high prevalence can be more important to public health that a rare risk factor with strong individual effect.

- PAF combines both aspects into a single statistic:
  Proportion of avoided cases if the individuals with X had been similar to those without X.

Population Attributable Fraction (PAF)

Cohort study

Levin (1953) was the first to proposed a statistic. A commonly used form is

$$\begin{aligned}\mathrm{PAR} &:= \frac{p(\mathrm{RR}-1)}{1+p(\mathrm{RR}-1)} = \frac{1}{1+\frac{1}{p(\mathrm{RR}-1)}} \\ &= \frac{1}{1+\frac{R_0}{p(R_1-R_0)}} = \frac{p(R_1-R_0)}{pR_1 - pR_0 + R_0} = \frac{p(R_1-R_0)}{pR_1-(1-p)R_0} \\ &= \frac{\text{Expected decrease in the cases}}{\text{Expected cases}}\end{aligned} \tag{4}$$

by writing $\mathrm{RR} = R_1/R_0$, the ratio of disease probabilities R_1 and R_1 among exposed and unexposed, respectively.
Note the the lhs of (4) does not depend on the *absolute* risks R_0 and R_1, only on their ratio, the RR!
RR can be estimated using e.g. Cox's proportional hazards model, and the prevalence p directly from the data.

Population Attributable Fraction (PAF)

Cross-sectional study

PAF can also be defined using e.g. logistic regression model for outcome Y_i.
Let X_i be a vector of m covariates, and X_i^* a modified version, in which the risk factor of interest has been set to state "unexposed".
Absolute risk R_i (and R_i^*) is based on X_i (X_i^*) and the regression coefficients β:

$$R_i := \mathbb{P}\{\, Y_i = 1 \mid X_i, \beta\,\} = \mathrm{expit}(X_i\beta) := \frac{1}{1+\exp\{-X_i\beta\}}. \tag{5}$$

Expected proportion of cases is the average of terms (5) called predictive margin[1]:

$$\mathrm{PM} := \frac{1}{n}\sum_{i=1}^{n} R_i \qquad \mathrm{PM}^* := \frac{1}{n}\sum_{i=1}^{n} R_i^*$$

PAF is defined as

$$\mathrm{PAF} := 1 - \frac{\mathrm{PM}^*}{\mathrm{PM}}$$

[1] Graubard and Korn (1999), Biometrics