Matematiikan ja tilastotieteen laitos
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Harjoitus 9
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In the first three exercises $f : [0,1] \to \mathbb{R}$ is a continuous function. Continuity is not always necessary, but it helps in some details.

Recall that f is of bounded variation if

$$V_f([0,1]) := \sup\{\sum_{i=1}^{k} |f(x_{i+1} - f(x_i)| : 0 = x_1 < x_2 < \cdots < x_{k+1} = 1\} < \infty.$$

Recall also that f is of bounded variation if and only if it is a difference of two increasing functions.

1. Show that if f is of bounded variation, then there is a signed measure $\mu_f : Bor([0,1]) \to [0,\infty)$ such that

$$\mu_f([a,b]) = f(b) - f(a) \text{ for all } 0 \leq a \leq b \leq 1.$$

2. Show that if f is the Cantor function (Devil's staircase), then μ_f and Lebesgue measure are mutually singular.

3. Show that if $\mu : Bor([0,1]) \to \mathbb{R}$ is a signed measure, then $f : x \mapsto \mu([0,x]), x \in [0,1]$, is of bounded variation and $V_f([0,1]) = V(\mu,[0,1])$.

4. Let $\emptyset \neq A \subset X$, (X,d) is a metric space. Show that $x \mapsto d(x,A)$ is a Lipschitz function.

5. Let $f : A \to \mathbb{R}$ be Lipschitz, where $A \subset X$. Define $g : X \to \mathbb{R}$,

$$g(x) = \inf\{f(y) + \text{Lip}(f)d(x,y) : y \in A\}, x \in X.$$

Show that g is Lipschitz, $\text{Lip}(g) = \text{Lip}(f)$ and $g(x) = f(x)$, when $x \in A$.

6. Let $C \subset [0,1]$ be the Cantor 1/3 set. Construct a Lipschitz function $f : \mathbb{R} \to \mathbb{R}$, which is not differentiable at any point of C.