Matematiikan ja tilastotieteen laitos
Reaalianalyysi II
Harjoitus 7
12.11.2015

In the following μ and ν are Radon measures in $\mathbb{R}^n$.

1. Prove that if $\mu(B) = \nu(B)$ for all closed balls $B \subset \mathbb{R}^n$, then $\mu = \nu$.

2. Prove that if μ is absolutely continuous with respect to ν, then

$$\int D_\nu\mu(x)^2 d\nu x = \int D_\nu\mu(x) d\mu x$$

.

3. Prove that $\underline{D}_\nu\mu(x) < \infty$ for μ almost all $x \in \mathbb{R}^n$, if and only if μ is absolutely continuous with respect to ν.

4. Prove that $\overline{D}_\nu\mu(x) = \infty$ for μ almost all $x \in \mathbb{R}^n$, if and only if μ and ν are mutually singular.

5. Let $1 < p \leq \infty$ and $f_i \in L^p(\mu), i = 1, 2, \ldots,$ be such that $f_i \geq 0$ and $\sup_i \int f_i^p d\mu < \infty$. Define

$$\mu_i(E) = \int_E f_i d\mu,$$

when E is μ measurable. Show that if the sequence (μ_i) converges weakly to a Radon measure ν, then ν is absolutely continuous with respect to μ.

6. Show that the assertion of the previous exercise is false if $p = 1$.