Matematiikan ja tilastotieteen laitos
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Harjoitus 6
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In the following $\mathcal{D}_k, k \in \mathbb{Z}$, is the family dyadic cubes in $\mathbb{R}^n$, whose side-length is 2^{-k} and $\mathcal{D} = \cup_{k\in\mathbb{Z}}\mathcal{D}_k$ and μ is a Radon measure in $\mathbb{R}^n$. So $Q \in \mathcal{D}_k$ if for some $j_i \in \mathbb{Z}, i = 1, \dots, n$,

$$Q = \{x \in \mathbb{R}^n : (j_i - 1)2^{-k} \leq x_i < j_i 2^{-k}, i = 1, \dots, n\}.$$

1. Prove the dyadic Vitali's covering theorem: Let $A \subset \mathbb{R}^n$ be bounded and $\mathcal{V} \subset \mathcal{D}$ such that every point of A belongs to an arbitrarily small cube of $\mathcal{V}$. If $\epsilon > 0$, there are disjoint cubes $Q_1, Q_2, \dots \in \mathcal{V}$ for which

$$A \subset \bigcup_i Q_i \text{ ja } \sum_i \mu(Q_i) \leq \mu(A) + \epsilon.$$

2. Prove the dyadic density theorem: if $A \subset \mathbb{R}^n$, then

$$\lim_{k\to\infty} \frac{\mu(A \cap Q_k(x))}{\mu(Q_k(x))} = 1$$

for μ almost all $x \in A$, where $Q_k(x)$ is that cube of $\mathcal{D}_k$ which contains x.

3. What are the smallest numbers $P(1)$ and $Q(1)$ for which Besicovitch's covering theorem holds in $\mathbb{R}$?

4. Show that the smallest numbers $P(n)$ and $Q(n)$ for which Besicovitch's covering theorem holds in $\mathbb{R}^n$ tend to ∞ as n tends to ∞.

5. Show that Besicovitch's covering theorem does not hold in any infinite-dimensional inner product space.

6. A metric space X is finite-dimensional if there exists $N \in \mathbb{N}$ such that for all $x \in X$ and $r > 0$ there are $x_1, \dots, x_N \in X$ for which

$$B(x, 2r) \subset \bigcup_{i=1}^{N} B(x_i, r).$$

Show that if there is a doubling Borel measure μ in X, then X is finite-dimensional. A measure μ is doubling if there is $C < \infty$ such that $0 < \mu(B(x, 2r)) \leq C\mu(B(x, r)) < \infty$ for all $x \in X$ an $r > 0$.