Matematiikan ja tilastotieteen laitos
Reaalianalyysi II
Exercise 4
15.10.2015

1. Prove that if $A \subset \mathbb{R}^m$ and $B \subset \mathbb{R}^n$ are compact, then (dim is Hausdorff dimension)

$$\dim A + \dim B \leq \dim(A \times B).$$

2. Prove for the Riesz capacities $C_s, s > 0$, without using the relations with Hausdorff measures, that
(a) $C_s(\{x\}) = 0$ for all $x \in \mathbb{R}^n$,
(b) $C_s(A) < \infty$, when $A \subset \mathbb{R}^n$ is bounded.

3. Prove that if $A \subset \mathbb{R}^n$ and $s > 0$, then $C_s(A) > 0$ if and only if there is $\mu \in \mathcal{M}_1(A)$ such that the potential $x \mapsto \int |x-y|^{-s}\, d\mu y$ is bounded.

4. Prove that if $K_i \subset \mathbb{R}^n, i = 1, 2, \ldots$, are compact sets such that $C_s(K_i) = 0$, then $C_s(\cup_i K_i) = 0$.

5. Let $\mu_j, j \in \mathbb{N}$, and μ be Radon measures in $\mathbb{R}^n$ such that $\mu_j \to \mu$ weakly. Prove that for all $s > 0$,

$$I_s(\mu) \leq \liminf_{j \to \infty} I_s(\mu_j).$$

6. What is the Hausdorff dimension of $\mathbb{R}$ when it is metrized by $d, d(x,y) = \sqrt{|x-y|}$?