Matematiikan ja tilastotieteen laitos
Reaalianalyysi II
Exercise 4
10.10.2015

1. Let $\Lambda : C_0(\mathbb{R}^n) \to \mathbb{R}$ be a positive linear functional. For any compact $K \subset \mathbb{R}^n$ let

$$C_0(K) = \{f \in C_0(\mathbb{R}^n) : \text{supp}(f) \subset K\}.$$

Prove that the restriction of Λ to $C_0(K)$ is bounded, that is, there is $C_K < \infty$ such that

$$|\Lambda(f)| \leq C_K \max_{x \in K} |f(x)| \text{ for all } f \in C_0(K).$$

2. Let $f_j \in L^1(\mathbb{R}^n), f_j \geq 0, j = 0, 1, 2, \dots,$ be such that $f_j \to f_0$ in $L^1(\mathbb{R}^n)$. Define $\mu_j(A) = \int_A f_j dm_n$, when $A \subset \mathbb{R}^n$ is Lebesgue measurable. Show that $\mu_j \to \mu_0$ weakly.

3. Let $g_j : \mathbb{R}^n \to \mathbb{R}, j = 1, 2, \dots,$ be non-negative continuous functions such that $\text{supp}(g_j) \subset B(0, 1/j)$ and $\int g_j dm_n = 1$. Show that if μ is a Radon measure on $\mathbb{R}^n$ (that is, in the σ-algebra of Borel subsets of $\mathbb{R}^n$), then the measures μ_j,

$$\mu_j(A) = \int_A \mu * g_j dm_n, A \in Bor(\mathbb{R}^n),$$

converge weakly to μ. The convolution $\mu * f$ is defined by

$$\mu * f(x) = \int f(x - y) d\mu y.$$

In the following X is a compact subset of $\mathbb{R}^n$ and $\mathcal{M}$ is the set of all Radon probability (that is, $\mu(X) = 1$) measures of X.

4. Show that d,

$$d(\mu, \nu) = \sup\{|\int f d\mu - \int f d\nu| : f : X \to \mathbb{R} \text{ 1-Lipschitz}\},$$

is a metric in $\mathcal{M}$. (f is 1-Lipschitz if $|f(x) - f(y)| \leq |x - y|$ for all $x, y \in X$.)

5. Show that if $\mu_j, \mu \in \mathcal{M}$ and $d(\mu_j, \mu) \to 0$, then $\mu_j \to \mu$ weakly.

6. Show that if $\mu_j, \mu \in \mathcal{M}$ and $\mu_j \to \mu$ weakly, then $d(\mu_j, \mu) \to 0$.