Matematiikan ja tilastotieteen laitos
Reaalianalyysi II
Exercise 3
1.10.2015

No lectures nor exercise sessions on September 23-24.

In the following μ is an outer on a set X and ν is an outer on a set Y. During the lectures $\mu \times \nu$ was defined for $E \subset X \times Y$ by

$$\mu \times \nu(E) = \inf\{\sum_{i=1}^{\infty} \mu(A_i)\nu(B_i) : E \subset \cup_{i=1}^{\infty} A_i \times B_i, A_i \ \mu\text{−measurable}, B_i \ \nu\text{−measurable}\}.$$

1. Prove that $\mu \times \nu$ is an outer measure.

2. Prove that $\mu \times \nu$ is metric if μ and ν are metric.

3. Is it true that $A \subset X$ is μ measurable and $B \subset Y$ is ν measurable whenever $A \times B$ is $\mu \times \nu$ measurable?

4. Let $X = Y = [0, 1]$ and let μ be the Lebesgue measure on X and ν the counting measure Y. Consider the diagonal $\{(x, x) : 0 \leq x \leq 1\}$ and show that the conclusion of Fubini's theorem fails in this case. Why is that?

5. Let $E \subset \mathbb{R}^n$ be Lebesgue measurable and $f : E \to \mathbb{R}$. Prove that f is a Lebesgue measurable function if and only if $\{(x, y) \in \mathbb{R}^{n+1} : x \in E, f(x) \geq y\}$ is a Lebesgue measurable set.

6. Let $E \subset \mathbb{R}^2$ be such that $E \cap F \neq \emptyset$ whenever $F \subset \mathbb{R}^2$ is closed with $m_2(F) > 0$. Suppose that no three different points of E lie on the same line. Prove that E is not Lebesgue measurable.

Such a set E exists, at least if believe in the continuum hypothesis. A hint for this can be found in Bruckner, Bruckner and Thomson: Real Analysis, Problem 6.4.12.