Matematiikan ja tilastotieteen laitos
Reaalianalyysi II
Exercise 2
17.9.2013

1. Show that in the definition of $\mathcal{H}^s(A)$ we can use open sets covering A without changing the value of $\mathcal{H}^s(A)$.

2. Define the spherical Hausdorff measure $\mathcal{S}^s$ in the same way as the Hausdorff measure but using balls as the covering sets. Show that for all $A \subset X$ we have

$$\mathcal{H}^s(A) \leq \mathcal{S}^s(A) \leq 2^s \mathcal{H}^s(A).$$

3. Show that for all $0 < s < \infty$ and for all $0 < \delta \leq \infty$, $\mathcal{H}^s(A) = 0$ if and only if $\mathcal{H}^s_\delta(A) = 0$.

4. Let ν be the counting measure on a metric space X; $\nu(A)$ is the number of elements in A, possibly ∞. Prove that $\nu = \mathcal{H}^0$. In particular ν is Borel regular, check this without referring to Hausdorff measures. When is ν a Radon measure?

5. Prove that if μ is a regular outer measure in a set X and $A_1, A_2, \ldots$ are arbitrary subsets of X such that $A_i \subset A_{i+1}$ for all i, then

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \lim_{i\to\infty} \mu(A_i).$$

6. Let $\mu : \mathcal{M} \to [0, \infty]$ be a measure on a σ-algebra $\mathcal{M}$ of a set X and define

$$\nu(A) = \inf\{\mu(B) : A \subset B, B \in \mathcal{M}\} \text{ for } A \subset X.$$

Prove that ν is a regular outer measure for which sets in $\mathcal{M}$ are measurable and $\nu(A) = \mu(A)$ for $A \in \mathcal{M}$.