Matematiikan ja tilastotieteen laitos
Real analysis II
Exercise 1
10.9.2015

1. Let μ be an outer measure in a metric space X. Prove that if Borel sets are μ measurable, then μ is metric.

2. Olkoon μ be an outer measure in a set X and let $f : X \to Y$ be an arbitrary function. Define the image measure (push forward) setting $f_{\#}\mu(A) = \mu(f^{-1}(A))$, when $A \subset Y$. Show that $f_{\#}\mu$ is an outer measure such that $A \subset Y$ is $f_{\#}\mu$ measurable whenever $f^{-1}(A)$ is μ measurable. In particular, $f_{\#}\mu$ is a Borel outer measure if X and Y are metric spaces, μ is Borel outer measure and f is continuous. What problems does the definition $f^{\#}\nu(A) = \nu(f(A))$ have? Here $A \subset X$ and ν is an outer measure in Y.

3. Let μ be an outer measure in a metric space X. Define the support of μ:

$$spt\mu = X \setminus \bigcup \{V : V \subset X \text{ open and } \mu(V) = 0\}.$$

Show that

$$spt\mu = \{x \in X : \mu(B(x, r)) > 0 \text{ for all } r > 0\}.$$

Show also that if X is separable (that is, X has a countable dense subset), then $spt\mu$ is the smallest closed set F, for which $\mu(X \setminus F) = 0$. Is this true without the separability assumption?

4. Give an example of a Borel outer measure μ in $\mathbb{R}$ for which $\mu(\mathbb{R}) = \mu(\mathbb{Q}) = 1$ and the support of μ is $\mathbb{R}$.

5. Let X and Y be separable metric spaces and $f : X \to Y$ continuous. Show that $f(spt\mu) \subset spt f_{\#}\mu$. Show also that if $spt\mu$ is compact, then $f(spt\mu) = spt f_{\#}\mu$.