Real Analysis II
9. exercise set, solutions

1. As f is of bounded variation it can be written as a difference of two increasing functions; $f = g - h$. We prove first that there is a Radon measure μ_g: $\mathrm{Bor}([0,1]) \longrightarrow [0,\infty[$ s.t.

$$\mu_g([a,b]) = g(b) - g(a) \quad \text{for all closed intervals} \quad [a,b] \subset [0,1]. \tag{1}$$

An obvious candidate for this measure is $\mu_g(A) = m_1(g(A))$, where m_1 the Lebesgue measure in $\mathbb{R}$, for all $A \in \mathrm{Bor}([0,1])$ but this definition leads to problems as g is not necessarily strictly increasing. Instead let $\widetilde{g}(x) = g(x) + x$ for all $x \in \mathbb{R}$. Now $\widetilde{g}$ is strictly increasing and we define $\mu_{\widetilde{g}}(A) = m_1(\widetilde{g}(A))$ for all $A \in \mathrm{Bor}([0,1])$. Now for all closed intervals $[a,b] \subset [0,1]$ we have

$$\mu_{\widetilde{g}}([a,b]) = g(b) - g(a) + b - a.$$

Thus by setting $\mu_g = \mu_{\widetilde{g}} - m_1$ we get the measure with property (1). Similarly we find a Radon measure $\mu_h([a,b]) = h(b) - h(a)$ for all closed intervals $[a,b] \subset [0,1]$.

Now define $\mu_f = \mu_g - \mu_h$. This is clearly a signed measure and for all closed intervals $[a,b] \subset [0,1]$ we have

$$\mu_f([a,b]) = (\mu_g - \mu_h)([a,b]) = [g(b) - h(b)] - [g(a) - h(a)] = f(b) - f(a),$$

as desired. $\square$

2. We need to find a measurable set C s.t $m_2(C) = 0 = \mu_f([0,1] \setminus C)$. Take C to be the standard Cantor set. It is well-known that $m_2(C) = 0$ (see e.g. Holopainen's lecture notes Reaalianalyysi I) so it is enough to show that $\mu_f([0,1] \setminus C) = 0$. But it is also well-known that $[0,1] \setminus C$ is a countable union of intervals and furthermore f is constant in each of these intervals. Consider any such interval J. Then $\mu_f(J) = \sup[f(b) - f(a) : [a,b] \subset J] = 0$ which shows that $\mu_f([0,1] \setminus C) = 0$ which finishes the proof. $\square$

3. As μ is a signed measure it has a Jordan decomposition $\mu = \mu^+ - \mu^-$, where μ^+ and μ^- are finite measures. Because of this, the functions $g(x) = \mu^+([0,x])$ and $h(x) = \mu^-([0,x])$ are increasing. Now $f = g - h$. By the remark in the exercise sheet f is of bounded variation.

We also have

$$\begin{aligned} V_f([0,1]) &= \sup_{0 \le a_1 < b_1 < a_2 < b_2 < \cdots < a_k < b_k \le 1} \sum |f(a_i) - f(b_i)| \\ &= \sup \sup_{0 \le a_1 < b_1 < a_2 < b_2 < \cdots < a_k < b_k \le 1} \sum |\mu([a_i, b_i]| \le V_f(\mu, [0,1]). \end{aligned}$$

To prove the opposite inequality, let $A_i \subset [0,1], i = 1, \ldots, m$, be disjoint Borel sets. We need to show that

$$\sum_{i=1}^{m} |\mu(A_i)| \le V_f([0,1]).$$

Let $\epsilon > 0$. Applying the approximation theorem for measures to μ^+ and μ^- we first find compact sets $K_i \subset A_i$ such that

$$\sum_{i=1}^{m} |\mu(A_i)| \le \sum_{i=1}^{m} |\mu(K_i)| + \epsilon.$$

Then we find disjoint open sets $U_i, i = 1, \ldots, m$, with $K_i \subset U_i$ and

$$\sum_{i=1}^{m} |\mu(K_i)| \le \sum_{i=1}^{m} |\mu(U_i)| + \epsilon.$$

Each U_i is a disjoint union of open intervals and by the compactness of K_i we can choose U_i so that it is a disjoint union of finitely many open intervals $(a_{i,j}, b_{i,j}), j = 1, \dots, m_i$. Then $|\mu((a_{i,j}, b_{i,j}))| = |f(a_{i,j}) - f(b_{i,j})|$ and we obtain

$$\begin{aligned}\sum_{i=1}^{m} |\mu(A_i)| &\le \sum_{i=1}^{m} |\mu(K_i)| + \epsilon \le \sum_{i=1}^{m} |\mu(U_i)| + 2\epsilon \\ &= \sum_{i=1}^{m} |\sum_{j=1}^{m_i} \mu((a_{i,j}, b_{i,j}))| + 2\epsilon \le \sum_{i=1}^{m}\sum_{j=1}^{m_i} |\mu((a_{i,j}, b_{i,j}))| + 2\epsilon \\ &= \sum_{i=1}^{m}\sum_{j=1}^{m_i} |f(a_{i,j}) - f(b_{i,j})| + 2\epsilon \le V_f([0,1]) + 2\epsilon.\end{aligned}$$

So $V_f(\mu, [0,1]) \le V_f([0,1])$. $\square$

4. Let $x, y \in X$. We have three cases to consider.

1°) Assume that $x \in A$. Then we have

$$|d(x,A) - d(y,A)| = d(y,A) = \inf_{z \in A} d(y,z) \le d(x,A).$$

2°) The case $y \in A$ is analogous to case 1°).

3°) Assume that $x, y \notin A$. By symmetry we can assume that $d(x,A) \ge d(y,A)$. Then we have

$$|d(x,A) - d(y,A)| = d(x,A) - d(y,A) = \inf_{z \in A} d(x,z) - \inf_{z \in A} d(y,z) \le d(x,y)$$

by simply applying triangle inequality.

Therefore $|d(x,A) - d(y,A)| \le d(x,y)$ for all $x, y \in X$ so $x \mapsto d(x,A)$ is 1-Lipschitz. $\square$

5. It follows easily from the Lipschitz-property of f that $f(x), g(x)$ are finite for every $x \in X$. To see that g is $\mathrm{Lip}(f)$-Lipschitz, let if $x_1, x_2 \in X$ and $\epsilon > 0$. Choose $y \in X$ such that $f(y) + \mathrm{Lip}(f) d(y, x_1) \le g(x_1) + \epsilon$. Then

$$g(x_2) - g(x_1) \le f(y) + \mathrm{Lip}(f)d(y,x_2) - (f(y) + \mathrm{Lip}(f)d(y,x_1) - \epsilon) \le \mathrm{Lip}(f)d(x_1,x_2) + \epsilon.$$

by the triangle inequality. This proves the first two claims.

If $x \in A$ we have

$$f(x) - \mathrm{Lip}(f)d(x,x) = f(x) \le f(y) + \mathrm{Lip}(f)d(x,y)$$

for every $y \in A$ by using the fact that f is $\mathrm{Lip}(f)$-Lipschitz. This gives $f = g$ in A. $\square$

6. Define $f(x) = d(x, C)$. Then f is Lipschitz by exercise 4. Let $x \in C$. For every $k = 1, 2, \dots$ there is a unique closed interval I_k of length 3^{-k} such that $x \in I_k$ and there is an open interval J_k of length 3^{-k} such that I_k and J_k have a common end-point $x_k \in C$ and $C \cap J_k = \emptyset$. Let y_k be the mid-point of J_k. Then $f(x) = f(x_k) = 0$ and $f(y_k) = 3^{-k}/2 \ge |x - y_k|/3$. Then both x_k and y_k tend to x, as $k \to \infty$, and it follows that the difference quotient $(f(y) - f(x))/(y - x)$ cannot have a limit as $y \to x$.