Real Analysis II
8. exercise set, solutions

1. Since μ is a signed measure we have $\mu(A) + \mu(B \setminus A) = \mu(B)$. If $\mu(B \setminus A) = -\infty$ then the left-hand side is not well defined. Thus $\mu(B \setminus A) \neq -\infty$ which immediately implies that $\mu(B) = \infty$ as $\mu(A) = \infty$. $\square$

2. Let $\mu = \mu^+ - \mu^- = \sigma^+ - \sigma^-$ be two Jordan decompositions of μ. Then μ^+ and μ^- are mutually singular as are σ^+ and σ^-. Therefore we find sets A and B s.t.

$$\mu^+(A) = \mu^-(X \setminus A) = 0 \quad \text{and} \quad \sigma^+(B) = \sigma^-(X \setminus B) = 0.$$

Then $X \setminus A$ is a positive set and B a negative set for μ. Hence for every $E \in \mathcal{M}$, $\mu((E \setminus A) \cap B)$ is both non-negative and non-positive, whence it is zero. Similarly $\mu((E \setminus B) \cap A) = 0$. This implies that $\mu(E \setminus A) = \mu(E \setminus B)$.

Now, for every $E \in \mathcal{M}$, we calculate

$$\begin{aligned}
\mu^+(E) &= \mu^+(E \setminus A) + \mu^+(A) \\
&= \mu^+(E \setminus A) \\
&= \mu(E \setminus A) + \mu^-(E \setminus A) \\
&= \mu(E \setminus A) \\
&= \mu(E \setminus B) \\
&= \mu(E \setminus B) + \sigma^-(E \setminus B) \\
&= \sigma^+(E \setminus B) \\
&= \sigma^+(E \setminus B) + \sigma^+(B) \\
&= \sigma^+(E).
\end{aligned}$$

Thus $\mu^+ = \sigma^+$. Similarly one proves that $\mu^- = \sigma^-$. Therefore the Jordan decomposition is unique. $\square$

3. Assume that $\mu \ll \nu$. Let $A \in M$ be s.t. $\nu(A) = 0$. Then by assumption $\mu(A) = 0$. Then $\mu(B) = 0$ for all $B \subset A$. But then $V(\mu, A) = 0$ so $V(\mu, \cdot) \ll \nu$.

For the other direction assume that $V(\mu, \cdot) \ll \nu$. Let $\nu(A) = 0$ for some $A \in M$. Then $|\mu(A)| \leq V(\mu, A) = 0$, which proves the claim. $\square$

4. Let m_n be the Lebesgue measure. This measure has no atoms. Suppose otherwise and let A its atom. Then by problem 5. there exists $a \in A$ s.t. $m_n(A \setminus \{a\}) = 0$. But then $m_n(A) = m_n(A \setminus \{a\}) + m_n(\{a\}) = 0 + 0 = 0$ which is not possible as A is an atom of m_n.

Let A be an atom of the counting measure. Clearly $\#A = 1$, as if $\#A \geq 2$ then A contains a singleton which has measure one. On the other hand, singletons are atoms because only subset they contain is the empty set which has measure zero.

5. Let split $\mathbb{R}^n$ into dyadic intervals. For $k \in \mathbb{Z}$ let $\mathcal{D}_k$ be the collection of dyadic cubes with sidelength 2^{-k}. Then we can split the measure as

$$\mu(A) = \sum_{Q \in \mathcal{D}_k} \mu(A \cap Q)$$

for all atoms A. Since A is an atom we find for all k an unique cube $Q_k \in \mathcal{D}_k$ s.t. $\mu(A) = \mu(A \cap Q_k)$. As $\mathbb{R}^n$ is complete $A \cap Q_k$ converges to some point $a \in \mathbb{R}^n$ as $k \longrightarrow \infty$. But then using the convergence of measures we have also $\mu(A \cap Q_k) \longrightarrow \mu(\{a\})$ as $k \longrightarrow \infty$. As $\mu(A \cap Q_k) = \mu(A)$ for all $k \in \mathbb{N}$ it follows that $\mu(A) = \mu(\{a\})$ i.e. $\mu(A \setminus \{a\}) = 0$ (because $\mu(A) < \infty$) as desired. $\square$

6. Since μ has no atoms, for every $F \in \mathcal{M}$ with $\mu(F) > 0$ there is $E \in \mathcal{M}, E \subset F$, such that $0 < \mu(E) < \mu(F)$. Then either $0 < \mu(E) \leq \mu(F)/2$ or $0 < \mu(F \setminus E) \leq \mu(F)/2$. Thus we find $E_1 \in \mathcal{M}$ such that $0 < \mu(E_1) \leq \mu(F)/2$. By the same argument there is $E_2 \in \mathcal{M}, E_2 \subset E_1$, such that $0 < \mu(E_2) \leq \mu(E_1)/2$. It follows that we can find sets $E \in \mathcal{M}, E \subset F$, with arbitrarily small positive measure. In particular, there is $A_1 \in \mathcal{M}$ such that $0 < \mu(A_1) \leq t$.

Suppose we have chosen $A_1 \subset A_2 \subset \ldots A_k$ such that $0 < \mu(A_j) \leq t$. Set

$$s_k = \sup\{\mu(B) : B \in \mathcal{M}, \mu(B) \leq t, A_k \subset B\}.$$

Choose $A_{k+1} \in \mathcal{M}$ such that $s_k - 1/k < \mu(A_{k+1}) \leq t$ and $A_k \subset A_{k+1}$. Let $A = \cup_{k=1}^{\infty} A_k$. Then $\mu(A) = \lim_{k\to\infty} \mu(A_k) \leq t$. Let $B \in \mathcal{M}, \mu(B) \leq t, A \subset B$. Since for all k, $A_k \subset B$, we have by the definition of s_k, $\mu(B) \leq s_k \leq \mu(A_{k+1}) + 1/k \leq \mu(B) + 1/k$, which implies $\mu(B) = \mu(A)$.

So we have shown that if $B \in \mathcal{M}, \mu(B) \leq t$ and $A \subset B$, then $\mu(B) = \mu(A)$. This implies that $\mu(A) = t$, since if we had $\mu(A) < t$, we could choose $E \subset X \setminus A$ with $0 < \mu(E) < t - \mu(A)$ and then $B = A \cup E$ would contradict the above statement.