Real Analysis II
7. exercise set, solutions

1. Let $\varepsilon > 0$ be arbitrary and $A \subset \mathbb{R}^n$ be Borel. It follows from Vitali's covering theorem that we find a countable cover $\{B_i\}$ of A consisting of closed balls s.t. $\mu(A) \le \sum \mu(B_i) \le \mu(A)+\varepsilon$ and $\nu(A) \le \sum \nu(B_i) \le \nu(A) + \varepsilon$. Then by assumption we have

$$\mu(A) \le \sum \mu(B_i) = \sum \nu(B_i) \le \nu(A) + \varepsilon.$$

As $\varepsilon > 0$ was arbitrary it follows that $\mu(A) \le \nu(A)$. Symmetric argument shows that $\nu(A) \le \mu(A)$, and it follows that $\mu = \nu$. □

2. As $D_\nu \mu$ is a Borel function there is an increasing sequence of Borel-measurable simple functions $f_i = \sum_j a_{ij}\chi_{A_{ij}}$ (sums are finite) s.t. $f_i \longrightarrow D_\nu\mu$ as $j \longrightarrow \infty$. As $\mu \ll \mu$, we have by Radon-Nikodym theorem that

$$\mu(A_{ij}) = \int D_\nu \mu \chi_{A_{ij}} \mathrm{d}\nu$$

for all i, j. Thus we can calculate

$$\int f_i \mathrm{d}\mu = \sum_j a_{ij}\mu(A_{ij}) = \sum_j a_{ij} \int D_\nu\mu\chi_{A_{ij}}\mathrm{d}\nu = \int D_\nu \mu f_i \mathrm{d}\nu.$$

By Monotone convergence theorem

$$\int D_\nu\mu \mathrm{d}\mu = \lim_{j\longrightarrow\infty} \int f_i \mathrm{d}\mu = \lim_{j\longrightarrow\infty} \int D_\nu \mu f_i \mathrm{d}\nu = \int D_\nu\mu^2 \mathrm{d}\nu,$$

as desired. □

3. Assume that μ is absolutely continuous wrt. ν. By Theorem 5.24. we have $\nu(\{x \in \mathbb{R}^n : \underline{D}_\nu\mu(x) = \infty\}) = 0$ so it follows from the absolute continuity that $\mu(\{x \in \mathbb{R}^n : \underline{D}_\nu\mu(x) = \infty\}) = 0$, as desired.

For the other direction, assume that $\underline{D}_\nu\mu(x) < \infty$ μ-a.e. $x \in \mathbb{R}^n$. Then $\mu(A) = 0$ if we set $A = \{x \in \mathbb{R}^n : \underline{D}_\nu\mu(x) = \infty\}$. Set also $A_k = \{x \in \mathbb{R}^n : \underline{D}_\nu\mu(x) \le k\}$ for all $k \in \mathbb{N}$. Let $B \subset \mathbb{R}^n$ be s.t. $\nu(B) = 0$. Now we have

$$\begin{aligned}\mu(B) &= \mu(A \cap B) + \mu\left(B \cap \bigcup A_k\right)\\ &= \mu\left(B \cap \bigcup A_k\right)\\ &\le \sum_i \ell\nu(B \cap A_i) = 0,\end{aligned}$$

for some $\ell \in \mathbb{N}$, where the penultimate set follows from Theorem 5.23. of Holopainen. This shows that μ is absolutely continuous wrt. ν. □

4. "Only if"-direction follows immediately from Theorem 5.30. of Holopainen's notes. For the other direction, assume $\overline{D}_\nu\mu(x) = \infty$ for a.e. $x \in \mathbb{R}^n$. Thus $\mu(\mathbb{R}^n \setminus \{x \in \mathbb{R}^n : \overline{D}_\nu\mu(x) = \infty\} = 0$. But by Theorem 5.24. of Holopainen, $\nu(\{x \in \mathbb{R}^n : \overline{D}_\nu\mu(x) = \infty\}) = 0$, so μ and ν are mutually singular. □

5. Let $A \subset \mathbb{R}^n$ be s.t. $\mu(A) = 0$ and $\epsilon > 0$. Then there is an open set V such that $A \subset V$ and $\mu(V) < \epsilon$. Let g_j be an increasing sequence of continuous compactly supported non-negative functions s.t. $g_j \longrightarrow \chi_V$ as $j \longrightarrow \infty$. Then by Dominated convergence theorem and Hölder's inequality we have

$$\begin{aligned}
\nu(A) \leq \nu(V) &= \int \chi_V \mathrm{d}\nu \\
&= \lim_{j \longrightarrow \infty} \int g_j \mathrm{d}\nu \\
&= \lim_{j \longrightarrow \infty} \lim_{i \longrightarrow \infty} \int g_j f_i \mathrm{d}\mu \\
&\leq \lim_{j \longrightarrow \infty} \limsup_{i \longrightarrow \infty} \left(\int g_j^{p/(p-1)} \right)^{1-\frac{1}{p}} \left(\int f_i^p \right)^{\frac{1}{p}} \\
&\leq \lim_{j \longrightarrow \infty} \limsup_{i \longrightarrow \infty} \mu(\mathrm{spt} g_j)^{1-\frac{1}{p}} \|f_i\|_p \\
&\leq \epsilon^{1-\frac{1}{p}} \sup_i \|f_i\|_p.
\end{aligned}$$

This shows that ν is absolutely continuous wrt. μ. □

6. Let μ be the Lebesgue measure on $\mathbb{R}$. Define for every $i \in \mathbb{N}$, the function $f_i : \mathbb{R} \longrightarrow \mathbb{R}$ by

$$f_i(x) = \begin{cases} i, & x \in [-1/2i, 1/2i] \\ 0, & \text{otherwise} \end{cases}$$

Then $f_i \geq 0$ for every $i \in \mathbb{N}$ and $\int f_i \mathrm{d}\mu = 1$, in particular, $\sup_i \int f_i < \infty$ and $f_i \in L^1(\mu)$ for every $i \in \mathbb{N}$. Also, as $\int f_i \mathrm{d}\mu = 1$ and $0 \in [-1/2i, 1/2i]$ for all $i \in \mathbb{N}$, it follows that μ_i converges to the Dirac measure δ_0. However, it is well-known that these measures are not absolutely continuous wrt. each other. □