Real Analysis II
6. exercise set, solutions

1. Let $\varepsilon > 0$ be arbitrary. Since μ is a Radon measure there exists an open set $U \subset \mathbb{R}^n$ s.t. $A \subset U$ and $\mu(U) \leq \mu(A) + \varepsilon$. For each $x \in A$ choose a maxmal cube in U containing x and contained in U. Call this cube Q_x. Now $\mathcal{B} = \{Q_x : x \in A\}$ is a cover of A. By maximality and the dyadic structure of A these cubes are pairwise disjoint. Also note that there are only countably many such cubes as there are only countably many dyadic cubes. Furthermore

$$\sum_i \mu(Q_i) = \mu\left(\bigcup_i Q_i\right) \leq \mu(U) \leq \mu(A) + \varepsilon,$$

which completes the proof. □

2. We imitate the proof of Theorem 5.14. of Holopainen's notes. As

$$\frac{\mu(A \cap Q_k(x))}{\mu(Q_k(x))} \leq 1$$

it suffices to show that

$$\liminf_{k\to\infty} \frac{\mu(A \cap Q_k(x))}{\mu(Q_k(x))} = 1$$

a.e. $x \in A$. Write

$$\left\{x \in A : \liminf_{k\to\infty} \frac{\mu(A \cap Q_k(x))}{\mu(Q_k(x))} < 1\right\} = \bigcup_{i=1}^{\infty} A_i,$$

where

$$A_i = \left\{x \in A : \liminf_{k\to\infty} \frac{\mu(A \cap Q_k(x))}{\mu(Q_k(x))} < 1 - \frac{1}{i}, |x| < i\right\}.$$

It is enough to show that A_i is measurable and that $\mu(A_i) = 0$ for every $i \in \mathbb{N}$.

Let $\varepsilon > 0$ be arbitrary. Since μ is a Radon measure we find an open set U s.t. $A_i \subset U$ and $\mu(U) \leq \mu(A_i) + \varepsilon$. Now the family

$$\mathcal{V} = \{Q_k(x) : x \in A_i, \mu(A \cap Q_k(x)) < (1 - 1/i)\mu(Q_k(x)), Q_k(x) \subset U\}$$

satisfies the conditions of Vitali's covering theorem. Using that we find pairwise disjoint dyadic cubes $Q_1, Q_2, ... \in \mathcal{V}$ s.t.

$$\mu\left(A_i \setminus \bigcup Q_k\right) = 0.$$

Now we estimate by using the measurability of A_i:

$$\begin{aligned}
\mu(A_i) &= \mu\left(A_i \cap \bigcup_k Q_k\right) + \mu\left(A_i \setminus \bigcup_k Q_k\right) \\
&= \mu\left(A_i \cap \bigcup_k Q_k\right) \\
&\leq \sum_k \mu(A_i \cap Q_k) \\
&\leq \left(1 - \frac{1}{i}\right) \sum_k \mu(Q_k) \\
&\leq \left(1 - \frac{1}{i}\right) \mu(U) \\
&\leq \left(1 - \frac{1}{i}\right) (\mu(A_i) + \varepsilon).
\end{aligned}$$

By letting $\varepsilon \longrightarrow 0$ we get that $\mu(A_i) \le (1-1/i)\mu(A_i) < \infty$. As $1-1/i < 1$ it follows that $\mu(A_i) = 0$. Finally A_i's are clearly measurable as they are unions of sets which are intersections of A with dyadic cubes. This concludes the proof. $\square$

3. Let $\mathcal{A}$ be a two point set $\{0,1\}$. By considering the cover $[-1/2,1/2]$, $[1/2,3/2]$ we see that $P(1), Q(1) \ge 2$. To prove that both $P(1)$ and $Q(1)$ can be equal to two we make the following inductive construction. Let $A \subset \mathbb{R}$ be a bounded set. By Besicovitch covering theorem we find a sequence of intervals $\{B_j\}$ which cover A and the number of overlappings is $P(1)$. Consider the interval B_1. If $A \subset \bigcup B_j \setminus \{B_1\}$, then set $B_1' = \emptyset$ and $B_1' = B_1$ otherwise. Assume that the set B_{n-1}' is chosen. Then set $B_n' = \emptyset$ if

$$A \subset \left(\bigcup_{j=1}^{n-1} B_i'\right) \cup \left(\bigcup_{j=n}^{\infty} B_i\right) \setminus \{B_n\}$$

and $B_n' = B_n$ otherwise.

This process produces a new set of intervals $B = \{B_j'\}$. Now $A \subset \bigcup B_j'$ as every point in A is covered by finitely many intervals. If some $x \in \mathbb{R}$ is covered by three intervals B_j then it follows from the construction that one of the new intervals is actually empty, because if any three intervals contain a common point, one of them is contained the union of the other two. Therefore $P(1) \le 2$ in this case. As noted in the beginning $P(1) \ge 2$ so $P(1)$ can be 2.

There is at most one interval $B_j' = B_{1,1}^+$ which intersects B_1' and contains its right end-point. Again there is at most one interval $B_j' = B_{1,2}^+$ which intersects $B_{1,1}^+$ and contains its right end-point. Continue this as long as such intervals are found. We put B_1' and the $B_{1,j}^+$ with even j into a collection $\mathcal{A}$ and the $B_{1,j}^+$ with odd j into a collection $\mathcal{B}$. Perform the similar operation the left of B_1 obtaining the intervals $B_{1,j}^-$ and putting them into $\mathcal{A}$ and $\mathcal{B}$ accordingly. If the families $\mathcal{A}$ and $\mathcal{B}$ do not yet cover A, let j be the smallest index for which B_j' is not in these families. Then B_j' must be disjoint with every interval in $\mathcal{A}$ and $\mathcal{B}$. Perform the same operation as above starting with B_j' and putting new intervals into $\mathcal{A}$ and $\mathcal{B}$. Continuing this as long as possible, perhaps infinitely many times, gives the desired disjoint families $\mathcal{A}$ and $\mathcal{B}$.

4. Let $e_1 = (1,0,...,0)$, $e_2 = (0,1,0,...,0)$, ..., $e_n = (0,...,0,1)$. Consider the set $\mathcal{A} = \{e_\ell\}_{\ell=1}^n$ and it's cover $\{\overline{B}(e_\ell,1)\}_{\ell=1}^n$. We have $|e_m - e_n| = \sqrt{2}$ for all m,n so every element of $\mathcal{A}$ belongs to exactly one of the cover sets. Furthermore the origin belongs to every cover set so we must have $P(n) \ge n$. Also $\overline{B}(e_i,1) \cap \overline{B}(e_j,1) \ne \emptyset$ for every i,j which implies that $Q(n) \ge n$. It follows immediately that $P(n), Q(n) \longrightarrow \infty$ as $n \longrightarrow \infty$. $\square$

5. Let $\mathcal{A} = \{e_n : n \in \mathbb{N}\}$ be an orthonormal basis in a given infinite dimensional inner product space. For all n,m we have

$$|e_n - e_m|^2 = \langle e_n - e_m, e_n - e_m\rangle = \langle e_n, e_n\rangle - \langle e_n, e_m\rangle - \langle e_n, e_m\rangle + \langle e_m, e_m\rangle = 2$$

so $|e_n - e_m| = \sqrt{2}$. Choose a cover $\{\overline{B}(e_n,1) : n \in \mathbb{N}\}$ for $\mathcal{A}$. As $1 < \sqrt{2}$ every element belongs to exactly one of the balls in the cover, so we cannot remove any of these sets. On the other-hand the origin belongs to every set in the cover. Thus Besicovitch's covering theorem does not hold for infinite dimensional inner product spaces. $\square$

6. Let $x \in X$ and $r > 0$. Let $\{x_j\}_{j=1}^N$ be the set of points in $B(x,2r)$ s.t. $d(x_i,x_j) > r$ whenever $i \ne j$. Here the number of points is a finite number depending on r. Clearly

$$B(x,2r) \subset \bigcup_{j=1}^N B(x_j,r).$$

Hence we are done if we show that N is uniformly bounded.

Note that balls $B(x_i, r/2)$ are disjoint and belong to $B(x_i, 3r)$. Thus

$$\sum_{j=1}^{N} \mu(B(x_j, r)) \leq \mu(B(x, 3r)).$$

On the other-hand, by using the observation that $B(x, 3r) \subset B(x_j, 6r)$ for every j and the doubling condition repeatedly we get

$$\begin{aligned}
\mu(B(x, 3r)) &\leq \mu(B(x_j, 6r)) \\
&\leq C\mu(B(x_j, 3r)) \\
&\leq C^2\mu(B(x_j, 3r/2)) \\
&\leq C^3\mu(B(x_j, 3r/4)) \\
&\leq C^3\mu(B(x_j, r)).
\end{aligned}$$

Thus

$$\sum_{j=1}^{N} \mu(B(x_j, r)) \leq C^3\mu(B(x_j, r)).$$

As this holds for every j and that $\mu(B(x, r)) > 0$ it follows that $N \leq C^3$. Proof completed.
□