Real Analysis II
5. exercise set, solutions

1. Let $s < \dim A$ and $t < \dim B$. Then by definition of Hausdorff measure $\mathcal{H}^s(A) > 0$ and $\mathcal{H}(B) > 0$. Then by Frostman's lemma we find Radon measures μ and ν s.t. $\operatorname{supp} \mu \subset A$, $\mu(A) > 0$, $\mu(B(x,r)) \leq cr^s$ for all $x \in \mathbb{R}^m$, $r > 0$ and $\operatorname{supp} \nu \subset B$, $\nu(B) > 0$, $\nu(B(y,r)) \leq c'r^t$ for all $y \in \mathbb{R}^n$, $r > 0$. As μ and ν are Radon measures we can define the product measure $\sigma := \mu \times \nu$. Note that $B((x,y),r) \subset B(x,r) \times B(y,r)$, so $\sigma(B((x,y),r)) \leq \mu(B(x,r))\nu(B(y,r)) \leq cc'r^{s+t}$. As $\operatorname{supp} \mu \subset A$ and $\operatorname{supp} \nu \subset B$ it follows that $\operatorname{supp} \sigma \subset A \times B$. Also clearly $\sigma(A \times B) = \mu(A)\nu(B) > 0$.

Hence by the other direction of Frostman's lemma $\mathcal{H}^{s+t}(A \times B) > 0$. This gives $\dim(A \times B) \geq s + t$. As s and t were arbitrary it follows that $\dim A + \dim B \leq \dim(A \times B)$. □

2. a) Assume otherwise: there exists $x \in \mathbb{R}^n$ s.t. $C_s(\{x\}) > 0$. Then by remark 4.76.3 there exists a probability measure $\mu \in \mathcal{M}_1(\{x\})$ s.t. $I_s(\mu) < \infty$. But clearly the only such probability measure is the Dirac measure δ_x. But then

$$I_s(\mu) = I_s(\delta_x) = \frac{1}{|x-x|^s} = \infty,$$

which is a contradiction. □

b) As A is bounded we have that diam $A < \infty$. But then $|x-y|^s \leq [\text{diam } (A)]^s$ for all $x, y \in A$. Thus for all $\mu \in \mathcal{M}_1(A)$ we have

$$I_s(\mu) \geq \frac{1}{[\text{diam } (A)]^s}$$

which gives $C_s(A) \leq [\text{diam } (A)]^2 < \infty$ as desired. □

3. Suppose $C_s(A) > 0$. Then there exists a probability measure $\mu \in \mathcal{M}_1(A)$ s.t. $I_s(\mu) < \infty$. Then $\int |x-y|^{-s}\, d\mu y < \infty$ for μ almost all x. It follows that there is $C < \infty$ such that the set $A = \{x : \int |x-y|^{-s}\, d\mu y < C\}$ has positive μ measure. Then A is Borel set and we can find a closed subset C of A with positive μ measure. Let ν be the restriction of μ to C. Then $\int |x-y|^{-s}\, d\nu y < C$ for $x \in C$. Since C is closed, for any $x \in \mathbb{R}^n$ there is $x' \in C$ such that $|x-y| \geq |x-x'|$ for all $y \in C$. If $y \in \mathbb{R}^n \setminus B(x, |x-x'|)$, then $|x'-y| \leq |x'-x| + |x-y| \leq 2|x-y|$. Since $\operatorname{supp}(\nu) \subset C$, we have

$$\int |x-y|^{-s}\, d\nu y = \int_{\mathbb{R}^n \setminus B(x,|x-x'|)} |x-y|^{-s}\, d\nu y \leq 2^s \int |x'-y|^{-s}\, d\nu y < 2^s C,$$

so $\int |x-y|^{-s}\, d\nu y$ is bounded.

If $\mu \in \mathcal{M}_1(A)$ s.t. $\int |x-y|^{-s}\, d\mu y$ is bounded, say $\int |x-y|^{-s}\, d\mu y$<C, then $I_s(\mu) \leq C$ and so $C_s(A) > 0$.

4. Suppose otherwise: $C_s(\cup K_i) > 0$. Then by remark 4.76.3 of Holopainen's notes there exists a measure $\mu \in \mathcal{M}_1(\cup K_i)$ s.t. $I_s(\mu) < \infty$. As $\mu \in \mathcal{M}_1(\cup K_i)$ we have $\mu(K_\ell) > 0$ for some $\ell \in \mathbb{N}$.

Let

$$\mu_\ell = \frac{1}{\mu(K_\ell)} \mu$$

be a normalized restriction of μ to K_ℓ. Now μ_ℓ is a Radon measure as $\mu(\mathbb{R}^n) = 1$ and the support of μ_ℓ is a subset of the compact set K_ℓ. The s-energy is

$$I_s(\mu_\ell) = \frac{1}{\mu(K_\ell)^2} \int_{K_\ell \times K_\ell} \frac{\mathrm{d}(\mu \times \mu)(x,y)}{|x-y|^s} \leq \frac{1}{\mu(K_\ell)^2} I_s(\mu) < \infty.$$

By remark 4.76.3 we have $C_s(K_\ell) > 0$ which is a contradiction. $\square$

5. We shall use the fact that $\mu_j \times \mu_j$ converges weakly to $\mu \times \mu$. This follows since the linear combinations of the functions of the form $(x,y) \mapsto f(x)g(y), f,g \in C_0(\mathbb{R}^n)$ are dense in $C_0(\mathbb{R}^n)$. This can be proven using the general Stone-Weierstrass approximation theorem or be a direct argument.

To apply the fact that $\mu_j \longrightarrow \mu$ weakly we need, as $|\cdot|^{-s} \notin C_0(\mathbb{R})$, we need to approximate it by $C_0(\mathbb{R})$ functions. To this end there is an increasing sequence of functions $\{f_k\}_{k=1}^{\infty}$ in $C_0(\mathbb{R})$ s.t. $0 \leq f_k(x) \leq |x|^{-s}$ for all $x \in \mathbb{R}^n, k \in \mathbb{N}$ and $\lim_{k\longrightarrow\infty} f_k(x) = |x|^{-s}$ for all $x \in \mathbb{R}$. Now we have

$$\begin{aligned} I_s(\mu) &= \sup_k \int_{\mathbb{R}^n\times\mathbb{R}^n} f_k(|x-y|)^{-s} \mathrm{d}(\mu\times\mu)(x,y) \\ &= \sup_k \lim_{j\longrightarrow\infty} \int_{\mathbb{R}^n\times\mathbb{R}^n} f_k(|x-y|)^{-s} \mathrm{d}(\mu_j\times\mu_j)(x,y) \\ &\leq \liminf{}_{j\longrightarrow\infty} \sup_k \int_{\mathbb{R}^n\times\mathbb{R}^n} f_k(|x-y|)^{-s} \mathrm{d}(\mu_j\times\mu_j)(x,y) \\ &\leq \liminf{}_{j\longrightarrow\infty} I_s(\mu_j), \end{aligned}$$

as desired. $\square$

6. The answer is $\dim(\mathbb{R},d) = 2$. Straightforward modification of the proof of Lemma 2.29.(i) of Holopainen's lecture notes shows that $\mathcal{H}^s(f(A)) \leq L^s \mathcal{H}^{\alpha s}(A)$ if f is a Hölder continuous mapping with exponent α and constant L for all $s > 0$. Let now $s > (1/\alpha)\dim A$. Then $\mathcal{H}^{\alpha s}(A) = 0$ so the above gives $\mathcal{H}^s(f(A)) = 0$. Then by the definition of the Hausdorff dimension gives $s > \dim f(A)$. This shows that

$$\dim f(A) \leq \frac{1}{\alpha} \dim A \tag{1}$$

for α-Hölder continuous maps f.

Consider first the identity mapping $id : \mathbb{R} \longrightarrow (\mathbb{R},d)$. This is Hölder continuous with exponent $\alpha = 1/2$ and constant $L = 1$. Thus by (1) we have

$$\dim(\mathbb{R},d) \leq \frac{1}{1/2} \dim \mathbb{R} = 2.$$

Consider then the identity mapping $id : (\mathbb{R},d) \longrightarrow \mathbb{R}$. This is Hölder continuous with exponent $\alpha = 2$ and constant $L = 1$. Thus by (1) we have

$$1 = \dim \mathbb{R} \leq \frac{1}{2} \dim(\mathbb{R},d)$$

so $\dim(\mathbb{R},d) \geq 2$. Thus $\dim(\mathbb{R},d) = 2$ as desired. $\square$