Real Analysis II
4. exercise set, solutions

1. By Riesz representation theorem the restriction of Λ to K must be of the form

$$\Lambda(f) = \int_K f(x)\mathrm{d}\mu(x)$$

for some Borel measure μ. Then we can estimate

$$|\Lambda(f)| = \left| \int_K f(x)\mathrm{d}\mu(x) \right| \leq \max_{x\in K} |f(x)| \cdot \int_K 1\mathrm{d}\mu.$$

Thus we may take

$$C_K = \int_K 1\mathrm{d}\mu,$$

which is finite as K is compact. □

A proof without Riesz representation theorem: Let $f \in C_0(\mathbb{R}^n)$ with $\mathrm{supp}(f) \subset K$. We may assume that f is non-negative. There exists a non-negative $h \in C_0(\mathbb{R}^n)$ such that $h = 1$ on K. Then $f \leq \|f\|h$ which implies $|T(f)| = T(f) \leq T(\|f\|h) = T(h)\|f\|$, and we can take $C_K = T(h)$.

2. First note that from the definition of μ_j and the fact that integrable functions can be approximated by simple functions, it follows that for any integrable function g we have

$$\int g\mathrm{d}\mu_j = \int g f_j \mathrm{d}m_n.$$

Let $h \in C_0(\mathbb{R}^n)$. Now we estimate

$$\begin{aligned} \left| \int h\mathrm{d}\mu_j - \int h\mathrm{d}\mu_0 \right| &= \left| \int h f_j \mathrm{d}m_n - \int h f_0 \mathrm{d}m_n \right| \\ &\leq \|h\|_\infty \cdot \left| \int (f_j - f_0)\mathrm{d}m_n \right|. \end{aligned}$$

The right-hand side goes to zero when j goes to infinity as $f_j \longrightarrow f_0$ in L^1 by assumption. As h was arbitrary it follows that $\mu_j \longrightarrow \mu_0$ weakly. □

3. As $\int g_j \mathrm{d}m_n = 1$ and μ is a Radon measure it follows by trivial estimation that $\mu_j(B(x,1))$ is finite for every $j \in \mathbb{N}$ and $x \in \mathbb{R}^n$. Thus μ_j is a Radon measure for every j. Note also that the support of $\int_{B(0,1)} |f(x-y) - f(x)| \cdot |g_j(y)|$ belongs to a compact set A. Now we may estimate by using Fubini's theorem:

$$\begin{aligned} \left| \int f\mathrm{d}\mu_j - \int f\mathrm{d}\mu \right| &= \left| \int f(y) \int g_j(x-y)\mu(x)\mathrm{d}y - \int f(x) \int g_j(y)\mathrm{d}y\mathrm{d}\mu(x) \right| \\ &= \left| \int \left(\int f(y)g_j(x-y)\mathrm{d}y - \int f(x)g_j(y)\mathrm{d}y \right) \mathrm{d}\mu(x) \right| \\ &= \left| \int \left(\int f(x-y)g_j(y)\mathrm{d}y - \int f(x)g_j(y)\mathrm{d}y \right) \mathrm{d}\mu(x) \right| \\ &= \left| \int \int_{B(0,1/j)} (f(x-y) - f(x))g_j(y)\mathrm{d}y\mathrm{d}\mu(x) \right| \\ &\leq \int \int_{B(0,1/j)} |f(x-y) - f(x)| \cdot |g_j(y)|\mathrm{d}y\mathrm{d}\mu(x). \end{aligned}$$

Let $\varepsilon > 0$ be arbitrary. As f is continuous we have that $|f(x) - f(y)| < \varepsilon$ when $|x - y|$ is small enough. Thus for large enough j it holds that

$$\int \int_{B(0,1/j)} |f(x-y) - f(x)| \cdot |g_j(y)| \mathrm{d}y \mathrm{d}\mu(x) < \varepsilon \int_A \int_{B(0,1/j)} |g_j(y)| \mathrm{d}y \mu(x) \leq \varepsilon \mu(A).$$

As $\varepsilon > 0$ was arbitrary and $\mu(A) < \infty$ it follows that $\int f \mathrm{d}\mu_j \longrightarrow \int f \mathrm{d}\mu$ so $\mu_j \longrightarrow \mu$. □

4. Let $\mu, \nu, \sigma \in \mathcal{M}$. Clearly $d(\mu, \nu) = d(\nu, \mu)$. Observe that by triangle inequality

$$\left| \int f \mathrm{d}\mu - \int f \mathrm{d}\nu \right| \leq \left| \int f \mathrm{d}\mu - \int f \mathrm{d}\sigma \right| + \left| \int f \mathrm{d}\sigma - \int f \mathrm{d}\nu \right|$$

for every 1-Lipschitz functions $f : X \longrightarrow \mathbb{R}$. Taking supremum over such f first on the right-hand side and then on the left-hand side gives $d(\mu, \nu) \leq d(\mu, \sigma) + d(\sigma, \nu)$.

If $d(\mu, \nu) = 0$, then $\int f \mathrm{d}\mu = \int f \mathrm{d}\nu$ for every 1-Lipschitz functions $f : X \longrightarrow \mathbb{R}$. Let K be a compact set. Then we can approximate the characteristic function of K by an increasing sequence $\{f_i\}$ of 1-Lipschitz functions (see Lemma 3.4. of Holopainen's notes). Then

$$\mu(K) = \lim \int f_i \mathrm{d}\mu = \lim \int f_i \mathrm{d}\nu = \nu(K).$$

It follows that $\mu(A) = \nu(A)$ for every Borel set A which gives $\mu = \nu$. Finally, if $\mu = \nu$, then $\int f \mathrm{d}\mu - \int f \mathrm{d}\nu = 0$ for every 1-Lipschitz functions $f : X \longrightarrow \mathbb{R}$ so $d(\mu, \nu) = 0$. Therefore d is a metric in $\mathcal{M}$. □

5. Let $f \in C_0(\mathbb{R}^n)$. By Weierstrass approximation theorem we find a sequence of polynomials $\{p_i\}$ s.t. $\|p_i - f\|_\infty \longrightarrow 0$ as $i \longrightarrow \infty$. Let $\varepsilon > 0$ be arbitrary and choose large enough j s.t.

$$\left| \int p_i \mathrm{d}\mu_j - \int p_i \mathrm{d}\mu \right| < \varepsilon.$$

As polynomials are Lipschitz in compact sets (actually boundedness is enough) it follows that for large enough i, j we get by the triangle inequality that

$$\left| \int f \mathrm{d}\mu_j - \int f \mathrm{d}\mu \right| \leq \left| \int f \mathrm{d}\mu_j - \int p_i \mathrm{d}\mu_j \right| + \left| \int p_i \mathrm{d}\mu_j - \int p_i \mathrm{d}\mu \right| + \left| \int f \mathrm{d}\mu - \int p_i \mathrm{d}\mu \right| < 3\varepsilon.$$

This shows that $\mu_j \longrightarrow \mu$ weakly. □

6. Suppose otherwise: there exists $\varepsilon > 0$ s.t. $d(\mu_j, \mu) \geq \varepsilon$. Thus for each j we find a 1-Lipschitz function $f_j : X \longrightarrow \mathbb{R}$ s.t.

$$\left| \int_j f_j \mathrm{d}\mu_j - \int f_j \mathrm{d}\mu \right| \geq \varepsilon.$$

By the Arzela-Ascoli theorem, see e.g. Bruckner, Bruckner and Thomson: Real Analysis, (f_j) has a subsequence, which we still denote by (f_j), which converges uniformly to some $f \in C_0(\mathbb{R}^n)$. Then

$$\left| \int f \mathrm{d}\mu_j - \int f \mathrm{d}\mu \right| \geq \left| \int f_j \mathrm{d}\mu_j - \int f \mathrm{d}\mu \right| - \left| \int f \mathrm{d}\mu_j - \int f_j \mathrm{d}\mu_j \right| \geq \varepsilon - \left| \int f \mathrm{d}\mu_j - \int f_j \mathrm{d}\mu_j \right|.$$

The last integrals tends to 0 by the uniform convergence. Thus $\mu_j \not\longrightarrow \mu$ weakly which is a contradiction. □