Real Analysis II
3. exercise set, solutions

1. Clearly $\mu \times \nu(\emptyset) = \mu(\emptyset)\nu(\emptyset) = 0 \cdot 0 = 0$. Suppose that $E \subset \bigcup_{k=1}^{\infty} E_k$ with $\mu \times \nu(E_k) < \infty$ for every k (if $\mu \times \nu(E_k) = \infty$ for some k there is nothing to prove). Let $\varepsilon > 0$ be arbitrary. For every k there exists a sequence of rectangles $\{A_j^k \times B_j^k\}_{j=1}^{\infty}$ with A_j^k μ-measurable and B_j^k ν-measurable s.t. $E_k \subset \bigcup_{j=1}^{\infty}(A_j^k \times B_j^k)$ and

$$\sum_{j=1}^{\infty} \mu(A_j^k)\nu(B_j^k) \leq \mu \times \nu(E_k) + \frac{\varepsilon}{2^k}.$$

Hence,

$$\begin{aligned}\mu \times \nu(E) &\leq \sum_{k=1}^{\infty}\sum_{j=1}^{\infty} \mu(A_j^k)\nu(B_j^k) \\ &\leq \sum_{k=1}^{\infty}\left(\mu \times \nu(E_k) + \frac{\varepsilon}{2^k}\right) \\ &= \varepsilon + \sum_{k=1}^{\infty} \mu \times \nu(E_k).\end{aligned}$$

As $\varepsilon > 0$ was arbitrary, it follows that $\mu \times \nu(E) \leq \sum_{k=1}^{\infty} \mu \times \nu(E_k)$. Thus $\mu \times \nu$ is an outer measure. □

2. Sorry, the claim is probably wrong without some extra assumptions on X and Y, PM. It is true if X and Y are separable. Then for any given $\epsilon > 0$, X and Y can be written as a countable union of sets of diameter less than ϵ. It follows that in the definition of $\mu \times \nu$ we can restrict to coverings with $A_i \times B_i$ for which $d(A_i \times B_i) < \epsilon$. Then it can be shown shown that $d(E, F) > 0$ implies $\mu \times \nu(E \cup F) = \mu \times \nu(E) + \mu \times \nu(F)$ as for Hausdorff measures. Under separability conditions one can also show that $\mu \times \nu$ is a Borel measure: then any open set is a countable union of product open sets, these are $\mu \times \nu$ measurable, so open sets are $\mu \times \nu$ measurable and therefore also Borel sets are $\mu \times \nu$ measurable. This is essentially the same as to say that for separable X and Y, the sigma-algebra generated by product open (or Borel) sets is the sigma-algebra of Borel sets of $X \times Y$.

3. The answer is **no**. To see this, choose arbitrary non-μ-measurable set $A \subset X$ and ν-measurable set $B \subset Y$ s.t. $\nu(B) = 0$. Let $\mu^*(A)$ be outer measure of A. Then $\mu \times \nu(A \times B) = \mu^*(A)\nu(B) = 0$ so $A \times B$ is $\mu \times \nu$-measurable.

4. Let us define a function $f : [0,1] \times [0,1] \longrightarrow \mathbb{R}$ by

$$f(x,y) = \begin{cases} 1, & x = y \\ 0, & \text{otherwise} \end{cases}$$

Now we have

$$\int_0^1 \left(\int_0^1 f(x,y)\mathrm{d}\mu(x)\right)\mathrm{d}\nu(y) = \int_0^1 0\mathrm{d}\nu(y) = 0$$

and

$$\int_0^1 \left(\int_0^1 f(x,y)\nu(y)\right)\mu(x) = \int_0^1 1\mathrm{d}\mu(x) = 1,$$

so the conclusion of Fubini's theorem does not hold. The reason is that the counting measure ν is not σ-finite on the interval $[0,1]$.

5. Assume that $f : E \longrightarrow \mathbb{R}$ is Lebesgue measurable function. Define $g : E \times \mathbb{R} \longrightarrow \mathbb{R}$ by $g(x,y) = f(x) - y$. Let us show that g is measurable. Write $h : \mathbb{R} \longrightarrow \mathbb{R}$, $h(y) = y$, $\widetilde{f}, \widetilde{g} : E \times \mathbb{R} \longrightarrow \mathbb{R}$, $\widetilde{f}(x,y) = f(x)$ and $\widetilde{h}(x,y) = h(y)$. Now $\widetilde{f}^{-1}([a,\infty[) = f^{-1}([a,\infty[) \times \mathbb{R}$ and $\widetilde{h}^{-1}([a,\infty[) = E \times h^{-1}([a,\infty[)$ for all $a \in \mathbb{R}$, which shows that $\widetilde{f}, \widetilde{g}$ are measurable functions as f, h are measurable functions and $E, \mathbb{R}$ are measurable sets. Therefore $g(x,y) = \widetilde{f}(x,y) - \widetilde{g}(x,y)$ is also measurable. Hence,

$$\{(x,y) \in \mathbb{R}^{n+1} : x \in E, f(x) \geq y\} = g^{-1}([0,\infty[).$$

As $[0,\infty[$ is a Borel set in $\mathbb{R}$ this shows the measurability.

For the other direction assume that the set $S = \{(x,y) \in \mathbb{R}^{n+1} : x \in E, f(x) \geq y\}$ is Lebesgue measurable. By Lemma 1.53. of Holopainen's lecture notes the set $\{x \in E : (x,y) \in S\}$ is measurable for every fixed $y \in \mathbb{R}$ which means that f is a Lebesgue measurable function. □

6. Assume that the set E is Lebesgue measurable. Then also the complement E^C is measurable. The first condition implies that E^C does not contain a closed set of positive measure. This implies that $m_2(E^C) = 0$, since it is well-known that a set of positive Lebesgue measure in $\mathbb{R}^2$ (more generally in a polish space) contains a closed set of positive Lebesgue measure. Let

$$\chi_E((x,y)) = \begin{cases} 1, & (x,y) \in E \\ 0, & (x,y) \notin E \end{cases}$$

The second condition implies that for a fixed $x \in \mathbb{R}$ there are at most two points $y \in \mathbb{R}$ s.t. $\chi_E(x,y) \neq 0$. Similarly, for fixed $y \in \mathbb{R}$, there is at most two points $x \in \mathbb{R}$ s.t. $\chi_E(x,y) \neq 0$. As E is measurable, also χ_E is. Now by Tonelli's theorem

$$m_2(E) = \int_{\mathbb{R}^2} \chi_E(x,y) \mathrm{d}m_2(x,y) = \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \chi_E(x,y) \mathrm{d}x \right) \mathrm{d}y = 0.$$

Let $A \subset \mathbb{R}^2$ be s.t. $m_2(A) > 0$. As E is measurable we have by Caratheodory's condition

$$0 < m_2(A) = m_2(A \cap E) + m_2(A \cap E^C) \leq m_2(E) + m_2(E^C) = 0 + 0 = 0,$$

which is a contradiction. Hence E is not Lebesgue measurable. □