Real Analysis II
2. exercise set, solutions

1. Let the Hausdorff measure given by open sets be $\widetilde{\mathcal{H}^s_\delta}(A)$. A δ-cover consisting of open sets is a δ-cover in usual sense, so $\widetilde{\mathcal{H}^s_\delta}(A) \geq \mathcal{H}^s_\delta(A)$. On the otherhand, let $\varepsilon > 0$ be arbitrary. For every δ-cover $\{A_i\}$ of the set A, we find an open cover $\{B_i\}$ s.t. $A_i \subset B_i$ and $d(B_i) \leq (1+\varepsilon)d(A_i)$ for every i. Hence $\widetilde{\mathcal{H}^s_{(1+\varepsilon)\delta}}(A) \leq \mathcal{H}^s_\delta(A)$. Letting $\varepsilon \longrightarrow 0$ gives $\widetilde{\mathcal{H}^s_\delta}(A) \leq \mathcal{H}^s_\delta(A)$. Combining this with the observation in the beginning of the proof we have $\widetilde{\mathcal{H}^s_\delta}(A) = \mathcal{H}^s_\delta(A)$, as desired. $\square$

2. Every δ-cover consisting of balls is a δ-cover in usual sense so $\mathcal{H}^s_\delta(A) \leq \mathcal{S}^s_\delta(A)$ for all $\delta > 0$. Letting $\delta \longrightarrow 0$ gives the left inequality.

On the other hand, for every δ-cover $\{A_i\}$ of A we can find 2δ-cover $\{B_i\}$ consisting balls s.t. $A_i \subset B_i$ and $\mathrm{diam}(B_i) \leq 2\mathrm{diam}(A_i)$. Then $\mathcal{S}^s_{2\delta}(A) \leq 2^s\mathcal{H}^s_\delta(A)$ and the right inequality follows by letting $\delta \longrightarrow 0$. $\square$

3. If $\mathcal{H}^s(A) = 0$, then by definition

$$0 \leq \mathcal{H}^s_\delta(A) \leq \sup_{\delta>0} \mathcal{H}^s_\delta(A) = \mathcal{H}^s(A) = 0$$

yielding $\mathcal{H}^s_\delta(A) = 0$.

For the opposite direction, suppose that $\mathcal{H}^s_\delta(A) = 0$. Let $0 < \varepsilon < \delta$ be arbitrary. Now, by the assumption $\mathcal{H}^s_\delta(A) = 0$, we can find a δ-cover $\{A_i\}$ of the set A s.t.

$$\sum_i d(A_i)^s < \varepsilon^s.$$

In particular, $d(A_i) < \varepsilon$ for every i, so our cover is actually an ε-cover. Letting $\varepsilon \longrightarrow 0$ gives $\mathcal{H}^s(A) = 0$. $\square$

4. Let us first consider the case where the set A is finite. In this case we have $A = \{a_1, ..., a_n\}$ for some $n \in \mathbb{N}$. Then for every $\delta > 0$ we have $A \subset \bigcup_{i=1}^n B(a_i, \delta/2)$. Thus

$$\mathcal{H}^0_\delta(A) \leq \sum_{i=1}^n (\mathrm{diam}\ B(a_i, \delta/2))^0 = n.$$

On the other hand if $0 < \delta < \min|a_i - a_j|$ we need at least n δ-sets to cover A. Thus $\mathcal{H}^0_\delta(A) \geq n$. Letting $\delta \longrightarrow 0$ gives $\mathcal{H}^0(A) = n$.

If A is infinite, it contains an n element subset for every $n \in \mathbb{N}$. Now the the finite case together with the monotonicity of Hausdorff measure yield $\mathcal{H}^0(A) = \infty$ in this case. Therefore $\mathcal{H}^0$ is the counting measure. $\square$

Let us then check that μ is Borel regular. It is of course a Borel measure. Let us first assume that $A \subset X$ is finite and write $A = \{a_1, ..., a_n\}$. As X is a metric space, A is closed and hence Borel. If A is infinite we have $\mu(A) = \mu(X)$ and we are done as X is open and hence Borel.

The necessary condition for measure being Radon measure is, by definition, that compact sets have finite measure. In this case it means that compact sets are finite i.e. X is discrete. Now it follows easily form Theorem 1.28. of Holopainen's notes that in this case the counting measure is a Radon measure (see Remark 1.30.)

5. Let $\mathcal{M} = \{E \subset X : E \text{ is } \mu - \text{measurable}\}$. Theorem 1.15. in Holopainen's lecture notes says that $\widetilde{\mu} := \mu|\mathcal{M}$ is a measure. By the regularity assumption for every A_n there exists

$\widetilde{\mu}$-measurable set B_n s.t. $A_n \subset B_n$ and $\mu(A_n) = \widetilde{\mu}(B_n)$. Write

$$B = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} B_k.$$

Clearly $\bigcup A_n \subset B$. Therefore

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \widetilde{\mu}(B) \overset{(*)}{=} \lim_{n \longrightarrow \infty} \widetilde{\mu}\left(\bigcap_{k=n}^{\infty} B_k\right) \leq \limsup_{n \longrightarrow \infty} \widetilde{\mu}(B_n) = \lim_{n \longrightarrow \infty} \mu(A_n),$$

where the equality (*) follows from Theorem 1.10. (a) of Holopainen's lecture notes. The reverse inequality is clear from the monotonicity of μ, so

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \lim_{n \longrightarrow \infty} \mu(A_n),$$

as desired. □

6. Let us first check that ν is an outer measure. Since $\mu \geq 0$ we have $\nu \geq 0$. Also $\nu(\emptyset) \leq \mu(\emptyset) = 0$ so $\nu(\emptyset) = 0$. It is clear from the definition that if $A \subset B$, then $\nu(A) \leq \nu(B)$. Let now $A \subset \bigcup_{i=1}^{n} A_i$. Assume that $\nu(A_i) < \infty$ since otherwise there is nothing to prove. Let $\varepsilon > 0$ be arbitrary. By definition we find a collection of μ-measurable sets B_i s.t. $A_i \subset B_i$ and $\nu(A_i) + \varepsilon 2^{-i} \geq \mu(B_i)$ for all $i = 1, ..., n$. Now we have

$$\nu\left(\bigcup_{i=1}^{n} A_i\right) \leq \mu\left(\bigcup_{i=1}^{n} B_i\right) = \sum_{i=1}^{n} \mu(B_i) \leq \varepsilon + \sum_{i=1}^{n} \nu(A_i).$$

As this holds for every $\varepsilon > 0$ we have

$$\nu(A) \leq \nu\left(\bigcup_{i=1}^{n} A_i\right) \leq \sum_{i=1}^{n} \nu(A_i).$$

Thus ν is an outer measure. □

Let us then check that ν is regular. Take a decreasing sequence of μ-measurable sets $\{B_n\}$ s.t. $A \subset B_n$ for each n and $\mu(B_n) \longrightarrow \nu(A)$. Let $B = \bigcap B_n$. Then B is μ-measurable and $A \subset B \subset B_n$ for each n so $\nu(A) \leq \mu(B) \leq \mu(B_n) \longrightarrow \nu(A)$ which shows that ν is regular once we have shown that sets in $\mathcal{M}$ are ν-measurable.

If $A \in \mathcal{M}$, the infimum is attained by $\mu(A)$, so $\nu(A) = \mu(A)$.

Let $A \in \mathcal{M}$. To prove that A is ν-measurable let $E \subset X$ be arbitrary. Let $\epsilon > 0$ and choose $B \in \mathcal{M}$ such that $E \subset B$ and $\mu(B) \leq \nu(E) + \epsilon$. Then

$$\nu(E) + \epsilon \geq \mu(B) = \mu(B \cap A) + \mu(B \setminus A) \geq \nu(E \cap A) + \nu(E \setminus A).$$

Letting $\epsilon \to 0$, we conclude that A is ν-measurable.