Real Analysis II
10. exercise set, solutions

1. First note that clearly $\emptyset, \mathbb{R}^n \in \tau_d$. Let $A, B \in \tau_d$. As A and B are Lebesgue measurable also $A \cap B$ is. We need to check that $A \cap B \in \tau_d$. Let $x \in A \cap B$. Then

$$m_n(A \cap B(x,r)) + m_n(B \cap B(x,r)) - m_n((A \cap B) \cap B(x,r)) \leq m_n(B(x,r))$$

or equivalently

$$\frac{m_n(A \cap B(x,r))}{m_n(B(x,r))} + \frac{m_n(B \cap B(x,r))}{m_n(B(x,r))} - 1 \leq \frac{m_n((A \cap B) \cap B(x,r))}{m_n(B(x,r))}$$

for all $r > 0$. Letting $r \longrightarrow 0$ in both sides shows that

$$\lim_{r \longrightarrow 0} \frac{m_n((A \cap B) \cap B(x,r))}{m_n(B(x,r))} \geq 1$$

when the limit exists. As the limit is clearly at most one it follows that it must be equal to one. Therefore $A \cap B \in \tau_d$. By iterating this result we get that all finite intersections belong to τ_d.

Let now $\mathcal{J}$ be an index set. Let $\{A_j\}_{j \in \mathcal{J}} \in \tau_d$ and $A = \cup_{j \in \mathcal{J}} A_j$. Suppose that A is bounded. Let $\epsilon > 0$, Then

$$\mathcal{B} = \{\overline{B}(x,r) : \exists j \in \mathcal{J} \text{ s.t. } m_n(A_j \cap \overline{B}(x,r)) > (1-\epsilon) m_n(\overline{B}(x,r))\}$$

satisfies the assumptions of Vitali's covering theorem. Hence there are disjoint $B_i \in \mathcal{B}, i = 1, 2, \ldots$, such that $m_n(A_{j_i} \cap B_i) > (1-\epsilon) m_n(B_i)$ for some $j_i \in \mathcal{J}$ and $m_n^*(A \setminus \cup_{i=1}^{\infty} B_i) = 0$. Let $E_\epsilon = \cup_{i=1}^{\infty} A_{j_i} \cap B_i$. Then

$$m_n(E_\epsilon) = \sum_{i=1}^{\infty} m_n(A_{j_i} \cap B_i) \geq \sum_{i=1}^{\infty} (1-\epsilon) m_n(B_i) \geq (1-\epsilon) m_n^*(A),$$

Since E_ϵ is measurable, $m_n^*(A) = m_n^*(E_\epsilon) + m_n^*(A \setminus E_\epsilon)$ and so $m_n^*(A \setminus E_\epsilon) \leq \epsilon m_n^*(A)$. Hence we can find measurable sets $E_{1/k} \subset A, k = 1, 2, \ldots$ such that $m_n^*(A \setminus E_{1/k}) \leq (1/k) m_n^*(A)$. Then $m_n^*(A \setminus \cup_{i=1}^{\infty} E_{1/k}) = 0$, which implies that A is measurable. If A is not bounded we can apply the above to the sets $A_j \cap B(0,k), j \in \mathcal{J}, k = 1, 2, \ldots$. Since every point of A clearly is a density point, we have showed that τ_d is indeed a topology. □

2. Suppose that $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is approximately continuous. Let $U \subset \mathbb{R}^n$ be open. For every $x \in f^{-1}(U)$ there is a measurable set $A_x \subset \mathbb{R}^n$ such that $D(A_x, x) = 1, x \in A_x$ and $f(A_x) \subset U$. Then by the same argument as in exercise 1, $A = \cup_{x \in f^{-1}(U)} A_x \in \tau_d$. Since $A = f^{-1}(U)$ it follows that f is continuous in the density topology.

Suppose then that f is continuous in the density topology. Then for every $k = 1, 2, \ldots$ there is $A_k \in \tau_d$ such that $f(A_k) \subset (f(x) - 1/k, f(x) + 1/k)$. There exist $r_k > 0$ such that

$$m_n(A_k \cap B(x,r)) > (1 - 1/k) m_n(B(x,r)) \text{ for } 0 < r < r_k.$$

We can recursively choose $0 < s_k < r_k$ such that

$$m_n(A_k \cap B(x,r) \setminus B(x, s_{k+1})) > (1 - 1/k) m_n(B(x,r)) \text{ for } s_k < r < r_k.$$

Then $A = \cup_{i=1}^{\infty} A_k \setminus B(x, s_{k+1})$ is a measurable for which $D(A, x) = 1$ and $f(y) \to f(x)$ when $y \to x, y \in A$. Thus f is approximately continuous at x.

3. Assume that $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is approximately continuous at the point $x \in \mathbb{R}^n$. Then, in particular, $\lim_{y\longrightarrow x, y\in A} f(y) = f(x)$ where A is a measurable for which $D(A, x) = 1$. Thus there exists $\delta > 0$ s.t. $|f(x) - f(y)| < \varepsilon$ when $y \in B(x, \delta) \cap A$. This gives

$$\lim_{r\longrightarrow 0} \frac{m_n(\{y \in B(x,r) : |f(y) - f(x)| < \varepsilon\})}{m_n(B(x,r))} = 1$$

or

$$\lim_{r\longrightarrow 0} \frac{m_n(\{y \in B(x,r) : |f(y) - f(x)| > \varepsilon\})}{m_n(B(x,r))} = 0, \qquad (1)$$

as desired.

For the converse direction suppose that (1) holds and let $x \in \mathbb{R}^n$. Let $\{A_k\}_{k\in\mathbb{N}}$ be a decreasing sequence of measurable sets in $\mathbb{R}^n$ defined by $A_k := \{y \in \mathbb{R}^n : |f(x)-f(y)| < 1/k\}$ s.t. x is a density point of all these sets. Then there exists positive numbers $\beta_1, \beta_2, ...$ strictly converging to zero s.t. x is a density point of the set

$$A := \bigcup_{n=1}^{\infty} (A_n \setminus B(x, \beta_n)).$$

Let $\{\gamma_k\}_{k\in\mathbb{N}}$ be a sequence of positive numbers less than one converging to zero. For each $k \in \mathbb{N}$ there exists $\delta_k > 0$ s.t. for all $d \in]0, \delta_k[$ we have

$$\frac{m_n(A_k \cap B(x,d))}{m_n(B(x,d))} > 1 - \gamma_k.$$

Choose a decreasing sequence $\{d_k\}_{k\in\mathbb{N}}$ convergent to zero s.t. $d_k \in]0, \delta_k[$ for each $k \in \mathbb{N}$. Then for $d_{k+1} \leq d \leq d_k$,

$$\begin{aligned}\frac{m_n(A \cap B(x,d))}{m_n(B(x,d))} &\geq \frac{m_n(A_k \cap B(x,d)) - m_n(B(x, \gamma_k d_{k+1}))}{m_n(B(x,d))} \\ &= \frac{m_n(A_k \cap B(x,d))}{m_n(B(x,d))} - (\gamma_k d_{k+1}/d)^n > 1 - 2\gamma_k.\end{aligned}$$

Now letting $d \longrightarrow 0$ shows that

$$\lim_{d\longrightarrow 0} \frac{m_n(A \cap B(x,d))}{m_n(B(x,d))} = 1$$

as $\gamma_k \longrightarrow 0$ when $d \longrightarrow 0$. This verifies the other direction. □

4. Assume first that $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is Lebesgue measurable. Then by Lusin's theorem there exists a sequence of closed sets $\{F_k\}_{k\in\mathbb{N}} \in \mathbb{R}^n$ s.t. the restriction of f to each F_k is continuous and $m_n(\mathbb{R}^n \setminus F_k) < 1/k$ for each $k \in \mathbb{N}$. Now f is approximately continuous for the set

$$\bigcup_{k\in\mathbb{N}} (F_k \cap D(F_k)),$$

where again $D(A)$ is the set of density points of the set A. Suppose then that f is approximately continuous for a.e. $x \in \mathbb{R}^n$. Let $y \in \mathbb{R}$ be arbitrary. Now it suffices to show that the set $B := \{x \in \mathbb{R}^n : f(x) < y\}$ is Lebesgue measurable. Let $C \subset \mathbb{R}^n$ be the set where f is approximately continuous. By Lebesgue density point theorem the set $B \setminus C$ has Lebesgue measure zero and hence it is a Lebesgue measurable set. Since $B = (B\cap C)\cup(B\setminus C)$ it suffices to show that $B \cap C$ is Lebesgue measurable. Let $x \in B \cap C$. Then by approximate continuity we find a Lebesgue measurable set A_x s.t. x is a density point of A_x and $\lim_{z\longrightarrow x, z\in A} f(z) = f(x)$. As $f(x) < y$ we can choose $A_x \subset B$ and also $A_x \subset C$ as $m_n(B \setminus C) = 0$. Thus

$$B \cap C = \bigcup_{x\in B\cap C} A_x,$$

which is Lebesgue measurable by the same argument we used in problem 1. to prove that arbitrary unions belong to the topology τ_d. This completes the proof. $\square$