Real Analysis II
1. exercise set, solutions

1. Let $A, B \subset X$ be arbitrary sets s.t. $d(A,B) > 0$. To prove that μ is metric, it suffices to show that $\mu(A \cup B) = \mu(A) + \mu(B)$. Let $\varepsilon = d(A,B)/2$. Let $C = \{x \in X : \operatorname{dist}(x, A) < \varepsilon\}$. Then C is a Borel set since it can be covered with balls $B(a, \varepsilon)$, $a \in A$. Furthermore, $(A \cup B) \cap C = A$ and $(A \cup B) \setminus C = B$, so by Caratheodory's condition

$$\begin{aligned}\mu(A \cup B) &= \mu((A \cup B) \cap C) + \mu((A \cup B) \setminus C) \\ &= \mu(A) + \mu(B).\end{aligned}$$

as desired. □

2. Let us first prove that $f_{\#}\mu$ is an outer measure. We clearly have $f_{\#}\mu(\emptyset) = \mu(f^{-1}(\emptyset)) = \mu(\emptyset) = 0$. Also, if $A \subset \bigcup A_i$, then

$$f_{\#}\mu(A) = \mu(f^{-1}(A)) \leq \sum_{i=1}^{\infty} \mu(f^{-1}(A_i)) = \sum_{i=1}^{\infty} f_{\#}\mu(A_i),$$

since μ is an outer measure and

$$f^{-1}(A) \subset f^{-1}\left(\bigcup_{i=1}^{\infty} A_i\right) = \bigcup_{i=1}^{\infty} f^{-1}(A_i).$$

This shows that $f_{\#}\mu$ is an outer measure. □

Next we show that $A \subset Y$ is $f_{\#}\mu$-measurable whenever $f^{-1}(A)$ is μ-measurable. Let $A \subset Y$ be s.t. $f^{-1}(A)$ is μ-measurable. Then for any $B \subset Y$ we have by Caratheodory's condition

$$\begin{aligned}f_{\#}\mu(B) = \mu(f^{-1}(B)) &= \mu(f^{-1}(B) \cap f^{-1}(A)) + \mu(f^{-1}(B) \setminus f^{-1}(A)) \\ &= \mu(f^{-1}(B \cap A)) + \mu(f^{-1}(B \setminus A)) \\ &= f_{\#}\mu(B \cap A) + f_{\#}\mu(B \setminus A).\end{aligned}$$

Hence A is measurable. □

Conditions imply that pre-images of Borel sets in Y are Borel in X, so $f_{\#}\mu$ is Borel.

The problem in defining $f^{\#}\nu(A) = \nu(f(A))$ is that we might have $f(A \cap B) \neq f(A) \cap f(B)$, so the above argument does not prove that $f(A) \subset Y$ is $f^{\#}\nu$-measurable whenever A is ν-measurable. Here is an explicit example: let ν be the Lebesgue measure on $[-1, 1]$ and $f(x) = x^2$. The $f^{\#}\nu([-1, 1]) = f^{\#}\nu([0, 1]) = f^{\#}\nu([-1, 0) = 1$, so intervals are not $f^{\#}\nu$ measurable and $f^{\#}\nu$ is not a Borel outer measure.

3. Let

$$\operatorname{spt}_1\mu = X \setminus \bigcup \{V : V \subset X \text{ open and } \mu(V) > 0\}$$

and

$$\operatorname{spt}_2\mu = \{x \in X : \mu(B(x,r)) > 0 \text{ for all } r > 0\}.$$

Suppose that $x \in \operatorname{spt}_1\mu$. If $x \notin \operatorname{spt}_2\mu$ then $\mu(B(x,r)) = 0$ for some $r > 0$, which obviously contradicts the definition of $\operatorname{spt}_1\mu$. Thus $\operatorname{spt}_1\mu \subset \operatorname{spt}_2\mu$.

Suppose next that $x \in \text{spt}_2\mu$. If $U \subset X$ is open, there exists $r' > 0$ s.t. $B(x, r') \subset U$. Then $\mu(U) \geq \mu(B(x, r')) > 0$ i.e. $x \in \text{spt}_1\mu$. Hence $\text{spt}_2\mu \subset \text{spt}_1\mu$.

Combining these considerations we have $\text{spt}_1\mu = \text{spt}_2\mu$, as desired. □

Let F be the smallest closed set s.t. $\mu(X \setminus F) = 0$. Then it follows immediately from the definition of support that $F \subset \text{spt}\mu$.

For the other direction recall that

$$\text{spt}\mu = \{x \in X : \mu(B(x, r)) > 0 \text{ for all } r > 0\}.$$

Since X is separable we can write

$$X \setminus \text{spt}\mu = \bigcup_{i=1}^{\infty} B(x_i, r_i)$$

where $\mu(B(x_i, r_i)) = 0$. Then

$$0 \leq \mu(X \setminus \text{spt}\mu) = \mu\left(\bigcup_{i=1}^{\infty} B(x_i, r_i)\right) \leq \sum_{i=1}^{\infty} \mu(B(x_i, r_i) = 0$$

so $\text{spt}\mu \subset F$. Hence, $\text{spt}\mu = F$.

Above the separability is needed for $\mu(X \setminus \text{spt}\mu) = 0$: let $X = \mathbb{R}$ be equipped with the discrete metric so that all sets open. Then the support of the Lebesgue measure is empty. □

4. Let us enumerate the rational numbers $\mathbb{Q} = \{q_1, q_2, ...\}$. For $A \subset \mathbb{R}$ we define

$$\mu(A) = \sum_{q_k \in A} 2^{-k}.$$

We claim that this is an outer measure satisfying the conditions of the problem. Let us first prove that it is indeed an outer measure. Clearly,

$$\mu(\emptyset) = \sum_{q_k \in \emptyset} 2^{-k} = 0.$$

Let $A \subset \bigcup A_i$. Then

$$\mu(A) = \sum_{q_k \in A} 2^{-k} \leq \sum_{q_k \in \bigcup A_i} 2^{-k} = \sum_{i=1}^{\infty} \sum_{g_k \in A_i} 2^{-k} = \sum_{i=1}^{\infty} \mu(A_i),$$

so μ is an outer measure.

To prove that μ is Borel we show that any set is measurable. Let $A \subset \mathbb{R}$ and $C \subset \mathbb{R}$ be arbitrary sets. Then every rational number in A belongs to exactly one of the sets $A \cap C$ and $A \setminus C$. Thus

$$\mu(A) = \sum_{q_k \in A} 2^{-k} = \sum_{q_k \in A \cap C} 2^{-k} + \sum_{q_k \in A \setminus C} 2^{-k} = \mu(A \cap C) + \mu(A \setminus C),$$

so by Caratheodory's condition C is μ-measurable. Thus μ is Borel.

Finally note that

$$\mu(\mathbb{R}) = \mu(\mathbb{Q}) = \sum_{k=1}^{\infty} 2^{-k} = 1.$$

Thus μ indeed works.

5. Recall that if X is separable, then sptμ (μ is an outer measure on X) is the smallest closed set F s.t. $\mu(X \setminus F) = 0$. Thus

$$0 = f_{\#}\mu(Y \setminus \mathrm{spt} f_{\#}\mu) = \mu(f^{-1}(Y \setminus \mathrm{spt} f_{\#}\mu)) = \mu(X \setminus f^{-1}(\mathrm{spt} f_{\#}\mu)).$$

As f is continuous, $f^{-1}(\mathrm{spt} f_{\#}\mu))$ is closed and we have $\mathrm{spt}\mu \subset f^{-1}(\mathrm{spt} f_{\#}\mu)$. It follows that

$$f(\mathrm{spt}\mu) \subset f(f^{-1}(\mathrm{spt} f_{\#}\mu)) \subset \mathrm{spt} f_{\#}\mu$$

as desired. □

Moreover, if sptμ is compact, then it follows from the continuity of f that $\mu(\mathrm{spt}\mu)$ is also compact. Then

$$0 \leq f_{\#}\mu(Y \setminus f(\mathrm{spt}\mu)) = \mu(f^{-1}(Y \setminus f(\mathrm{spt}\mu)) = \mu(X \setminus f^{-1}(f(\mathrm{spt}\mu))) \leq \mu(X \setminus \mathrm{spt}\mu) = 0.$$

Thus $f_{\#}\mu(Y \setminus f(\mathrm{spt}\mu)) = 0$. Since $f_{\#}\mu(Y \setminus \mathrm{spt} f_{\#}\mu) = 0$ and $\mathrm{spt} f_{\#}\mu$ is the smallest set having this property, it follows that $\mathrm{spt} f_{\#}\mu \subset f(\mathrm{spt}\mu)$. As we have proved that $f(\mathrm{spt}\mu) \subset \mathrm{spt} f_{\#}\mu$ it follows that $f(\mathrm{spt}\mu) = \mathrm{spt} f_{\#}\mu$. □