Raja-arvot, syksy 2015, kurssikoe 22.10.2015

Mallivastaukset (MK)

1. $$\lim_{n\to\infty} \frac{(n+2)(2n+3)}{(3n+4)(4n+5)} = \lim_{n\to\infty} \frac{2n^2+7n+6}{12n^2+31n+20} \qquad (+1p)$$

$$= \lim_{n\to\infty} \frac{n^2\left(2+\frac{7}{n}+\frac{6}{n^2}\right)}{n^2\left(12+\frac{31}{n}+\frac{20}{n^2}\right)} = \lim_{n\to\infty} \frac{2+\frac{7}{n}+\frac{6}{n^2}}{12+\frac{31}{n}+\frac{20}{n^2}} \qquad (+1p)$$

$$= \frac{\lim\limits_{n\to\infty} 2 + 7\cdot\lim\limits_{n\to\infty}\frac{1}{n} + 6\cdot\lim\limits_{n\to\infty}\frac{1}{n}\cdot\lim\limits_{n\to\infty}\frac{1}{n}}{\lim\limits_{n\to\infty} 12 + 31\cdot\lim\limits_{n\to\infty}\frac{1}{n} + 20\cdot\lim\limits_{n\to\infty}\frac{1}{n}\cdot\lim\limits_{n\to\infty}\frac{1}{n}} \qquad (+2p)$$

$$= \frac{2+7\cdot 0+6\cdot 0\cdot 0}{12+31\cdot 0+20\cdot 0\cdot 0} = \frac{2}{12} = \underline{\underline{\frac{1}{6}}}. \qquad (+1p)$$

2. Olk. $\varepsilon>0$. Valitaan $n_\varepsilon \in \mathbb{N}$, s.e. $n_\varepsilon > \frac{1}{\varepsilon}$. Kun $n \geq n_\varepsilon$, niin (+3p)

$$\left|\frac{n+3}{2n+5}-\frac{1}{2}\right| = \left|\frac{1}{4n+10}\right| = \frac{1}{4n+10} < \frac{1}{n} \leq \frac{1}{n_\varepsilon} < \varepsilon. \qquad (+3p)$$

Siis $\lim\limits_{n\to\infty} \frac{n+3}{2n+5} = \frac{1}{2}$. □

3. Olk. $\varepsilon>0$. Valitaan $\delta=\min\{1,\varepsilon\}$. Kun $0<|x-3|<\delta$, niin (+2p) (+1p)

$$\left|\frac{x+1}{x+2}-\frac{4}{5}\right| = \left|\frac{x-3}{5x+10}\right| = \frac{|x-3|}{5|x+2|} < \frac{|x-3|}{5(2+2)}$$

$$= \frac{|x-3|}{20} < |x-3| < \varepsilon. \qquad (+3p)$$

Näin ollen $\lim\limits_{x\to\infty} \frac{x+1}{x+2} = \frac{4}{5}$. □

4. (a) $x_{n+1} - x_n = \frac{1}{2}(1+x_n^2) - x_n = \frac{1}{2}(x_n^2 - 1)^2 \geq 0$,

joten jono (x_n) on kasvava. (+2p)

Os. induktiolla, että 1 on jonon (x_n) yläraja.

<u>Alkuaskel $n=1$:</u> $x_1 = 0 \leq 1$.

<u>Induktio-oletus:</u> Väite pätee arvolla $n=k$:

$$x_k = \frac{1}{2}(1+(x_{k-1})^2) \leq 1.$$

<u>Induktioaskel:</u> Väite pätee arvolla $n=k+1$:

$$x_{k+1} = \frac{1}{2}(1+x_k^2) = \frac{1}{2} + \frac{1}{2}x_k^2 \leq \frac{1}{2} + \frac{1}{2}\cdot 1^2$$
$$= \frac{1}{2} + \frac{1}{2} = 1.$$ (+2p)

Induktioperiaatteen nojalla väite pätee. □

(b) Koska jono (x_n) on kohdan (a) perusteella kasvava ja ylhäältä rajoitettu, niin se suppenee. □ (+1p)

Koska jono (x_n) suppenee, niin

$$a = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty}\left(\frac{1}{2}(1+x_n)^2\right) = \lim_{n\to\infty}\left(\frac{1}{2}+\frac{1}{2}x_n^2\right)$$
$$= \lim_{n\to\infty}\frac{1}{2} + \frac{1}{2}\lim_{n\to\infty} x_n \cdot \lim_{n\to\infty} x_n = \frac{1}{2} + \frac{1}{2}\cdot a \cdot a$$
$$= \frac{1}{2} + \frac{1}{2}a^2 \iff a = \frac{1}{2} + \frac{1}{2}a^2$$
$$\iff (a-1)^2 = 0$$
$$\iff a = 1.$$

Siis $\lim_{n\to\infty} x_n = 1$. (+1p)