UH Probability theory II, Autumn 2015, Exercises 9 (18.11.2015)

Option prices and their "Greeks" In mathematical finance, the Black and Scholes model a stock price $S_t(\omega)$ at time $t > 0$ by a log-normal distribution, (which in other words is the exponential of a Gaussian)

$$S_t(\omega) = S_0 \exp\biggl(\sigma \sqrt{t} G(\omega) - \frac{1}{2}\sigma^2 t \biggr)$$

where $S_0 > 0$ is the stock price at the present time $t = 0$ and $G(\omega) \sim \mathcal{N}(0,1)$ is a standard Gaussian distribution, with

$$P(G \le x) = \int_{-\infty}^{x} \phi(y) dy, \qquad \phi(y) = \frac{\exp(-y^2/2)}{\sqrt{2\pi}}$$

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An european option with maturity $t > 0$ is a function of the stock price $H(\omega) = h(S_t(\omega))$, where $x \mapsto h(x)$ is measurable.

It follows that the price of the option at the present time $t = 0$ is the expectation $c(t, S_0, \theta) := E_P(h(S_t))$ with respect to the pricing probability P.

1. Show that $E_P(S_t) = S_0$. For this reason, such P is called risk-neutral.

2. Assume first that $x \mapsto h(x)$ is differentiable and show that the option price $c(t, S_0, \sigma)$ is differentiable w.r.t. the parameters t, S_0 and σ. In order to justify the change of order in derivation and integration operators, show that the appropriate uniform integrability condition holds for the derivatives in a neighbourhood of the parameter vector.

3. Show that $c(t, S_0, \sigma)$ is differentiable with respect to the parameters t, S_0 and σ, also in situations where the function $h(x)$ in not differentiable, giving the suitable uniform integrability conditions for the derivatives.

   Hints write

$$c(t, S_0, \sigma) = \int_{\mathbb{R}} h\bigl(\exp\bigl(\log S_0 - \frac{1}{2}\sigma^2 t + \sigma\sqrt{t} y\bigr)\bigr)\phi(y) dy$$

$$= \frac{1}{\sigma\sqrt{t}} \int_{\mathbb{R}} h\bigl(\exp(x)\bigr)\ \phi\biggl(\frac{x - \log S_0 + \frac{1}{2}\sigma^2 t}{\sigma\sqrt{t}}\biggr) dx$$

using the change of variables

$$x = \log S_0 - \frac{1}{2}\sigma^2 t + \sigma\sqrt{t} y$$
$$y = \frac{x - \log S_0 + \frac{1}{2}\sigma^2 t}{\sigma\sqrt{t}} \; .$$

Recall also that the density of the standard Gaussian distribution satisfies $\frac{d}{dx}\phi(x) = -x\phi(x)$.

4. Compute the option price and $c(t, S_0, \sigma)$ and the sensitivity parameters

$$\frac{\partial c(t, S_0, \sigma)}{\partial t}, \; \frac{\partial c(t, S_0, \sigma)}{\partial S_0}, \; \frac{\partial c(t, S_0, \sigma)}{\partial \sigma} \; ,$$

(these derivatives are referred in the math-finance literature as "Greeks") in these two cases

(a)

$$h(S_1) = (S_1 - K)^+$$

which is an european call-option with strike price $K > 0$,

(b)

$$h(S_1) = (K - S_1)^+ = (S_1 - K)^- = K - S_1 - (S_1 - K)^+$$

which is an european put-option with strike price $K > 0$.

The european call-option $\big(S_t(\omega) - K\big)^+$ gives to the option holder the right (without obligation) to buy from his counterpart at the maturity time t one share at the pre-determined price K. If at the time of maturity t the market price of the stock $S_t(\omega)$ is higher than the strike price K, the option holder will exercise her right, buying on share at price K and selling in the market at market price $S_t(\omega)$, making a profit of $\big(S_t(\omega) - K\big)^+$. If at maturity time t $S_t(\omega) \le K$, the call-option becomes worthess, and the option holder does not make profit.

Analogously the european put-option $\big(S_t(\omega) - K\big)^-$ gives to the option holder the right (without obligation) to sell from the counterpart one share at the pre-determined price K.

The put-call parity between european put and call options is the equation

$$S_t(\omega) - K = (S_t(\omega) - K)^+ - (K - S_t(\omega))^+$$

Since the contract on the left side at time $t = 0$ has price $(S_0 - K)$, we have the following parity relation between the put and call option prices at time $t = 0$:

$$S_0 - K = c\big((S_t - K)^+\big) - c\big((K - S_t)^+\big)$$

For simplicity we have assumed that a riskless investment has zero-interest rate, equivalently all values are discounted and expressed in present-time values.