UH Probability Theory II, Fall 2015, Exercises 11 (2.12.2015)

In all problems the random variables live in the probability space $(\Omega, \mathcal{F}, P)$ and $\mathcal{G} \subseteq \mathcal{F}$ is a sub σ-algebra of $\mathcal{F}$.

1. For $X \in L^1(\Omega, \mathcal{F}, P)$, show that

$$\big|E_P\big(X\big|\mathcal{G}\big)(\omega)\big| \le E_P\big(\,|X|\,\big|\mathcal{G}\big)(\omega)\,.$$

2. Let $X, Y \in L^2(\Omega, \mathcal{F}, P)$ and $\mathcal{G} \subseteq \mathcal{F}$ be a sub-σ-algebra. The covariance between the random variables X and Y was defined as

$$\mathrm{Cov}_P(X, Y) = E_P(XY) - E_P(X)E_P(Y)$$

and we define now the conditional covariance between the random variables X and Y given the information in the σ-algebra $\mathcal{G}$ as the random variable

$$\mathrm{Cov}_P(X, Y|\mathcal{G})(\omega) = E_P(XY|\mathcal{G})(\omega) - E_P(X|\mathcal{G})(\omega)E_P(Y|\mathcal{G})(\omega)$$

Prove the following identity:

$$\mathrm{Cov}_P(X, Y) = E_P\big(\,\mathrm{Cov}(X, Y|\mathcal{G})\big) + \mathrm{Cov}_P\big(E_P(X|\mathcal{G}), E_P(Y|\mathcal{G})\big)$$

3. On a probability space $(\Omega, \mathcal{F}, P)$, let $A \in \mathcal{F}$ with $P(A) > 0$. For a random variable $X(\omega) \ge 0$ P-a.s., we have defined the (elementary) conditional expectation of X conditioned on the **event** A as the deterministic constant

$$E_P(X|A) = \frac{E_P(X\mathbf{1}_A)}{P(A)}$$

Consider now the sub σ-algebra $\mathcal{G} = \sigma(A) = \{\emptyset, \Omega, A, A^c\} \subseteq \mathcal{F}$.

   (a) Compute the conditional expectation of X conditioned on the σ-algebra $\mathcal{G}$ according to the Kolmogorov definition of $E_P(X|\mathcal{G})(\omega)$

   (b) Let $X(\omega)$ be a random variable with cumulative distribution $F_X(t) = P(X \le t)$, and let $Y(\omega) = \mathbf{1}_{(a,b]}(X(\omega))$.
   Compute the conditonal expectation $E_P(X|\mathcal{G})(\omega)$ with respect to the σ-algebra $\mathcal{G} = \sigma(Y)$.

   (c) Let $\tau(\omega) \ge 0$ a non-negative random variable with λ-exponential distribution such that $P_\lambda(\tau > s) = \exp(-\lambda s)$, for $s \ge 0$. We can interpret τ as a random time. For $t \ge 0$ compute the conditional expectation $E_P\big(\tau\big|\sigma(\tau \wedge t)\big)(\omega)$ where $\tau(\omega) \wedge t = \min\{\tau(\omega), t\}$.

4. Let $X_1, \ldots, X_n$ independent and identically distributed random variables, and $S_n = (X_1 + X_2 + \cdots + X_n)$. Compute $E_P(X_1|\sigma(S_n))(\omega)$.

   Hint: Note that

$$\sum_{k=1}^{n} E_P(X_k|\sigma(S_n))(\omega) = E_P(S_n|\sigma(S_n))(\omega) = S_n(\omega)$$

   (why?), justify the equalities and then use symmetry.

5. Let $X, Y \in L^2(\Omega, \mathcal{F}, P)$. Prove the Cauchy-Schwartz inequality for the conditional expectations:

$$\big\{E_P(XY|\mathcal{G})(\omega)\big\}^2 \le E_P(Y^2|\mathcal{G})(\omega)E_P(X^2|\mathcal{G})(\omega)$$

   Hint: follow the steps of the proof of the Cauchy-Schwartz inequality for expectation.

6. Prove the following conditional Chebychev inequality: when $X \in L^2(\Omega, \mathcal{F}, P)$ and $Y(\omega)$ is $\mathcal{G}$-measurable, with $P(Y(\omega) \ge 0) = 1$,

$$P(|X| > Y|\mathcal{G})(\omega) \le \frac{E_P\big(X^2|\ \mathcal{G})(\omega)}{Y^2(\omega)} \qquad P\text{-almost surely}$$

   P-almost surely. In particular it holds when $Y(\omega) \equiv y > 0$ is a deterministic constant.