UH Probability Theory II, exam Solutions (7.13.2016)

You can choose whether to write the exam in English or in Finnish, the Finnish version of the same exam is also available! The duration of the exam is 3 hours and 30 minutes. In the problems, all random variables are defined on a probability space $(\Omega, \mathcal{F}, P)$.

1. Show that for a random variable X with $X(\omega) \geq 0$ P-almost surely, we have

$$\int_0^\infty \frac{E_P(X \wedge t^2)}{t^2} dt = 2E\big(\sqrt{X}\big)$$

where $x \wedge t = \min\{x, t\}$. **Hint:**Use Fubini theorem.

Solution Let $F(t) = P(X \leq t)$ be the cumulative distribution function of the random variable X.

$$\int_0^\infty \frac{E_P(X \wedge t^2)}{t^2} dt = \int_0^\infty \int_0^\infty (x \wedge t^2) F(dx) t^{-2} dt =$$
$$= \int_0^\infty \int_0^{t^2} x F(dx) t^{-2} dt + \int_0^\infty \int_{t^2}^\infty t^2 F(dx) t^{-2} dt = \quad \text{Fubini}$$
$$= \int_0^\infty \int_{\sqrt{x}}^\infty t^{-2} dt x F(dx) + \int_0^\infty P(X > t^2) dt$$
$$= \int_0^\infty x^{-1/2} x F(dx) + \int_0^\infty P(\sqrt{X} > t) dt$$
$$= \int_0^\infty x^{-1/2} F(dx) + E_P(\sqrt{X}) = 2E_P(\sqrt{X})$$

2. Let $X(\omega)$ be a random variable, with $X(\omega) > 0$ (strictly) P-almost surely. Show that

$$E_P\big(1/X\big) \geq 1/E_P(X), \quad E_P(\log(X)) \leq \log E_P(X), \quad E_P(X \log(X)) \geq E_P(X) \log E_P(X).$$

Hint: which of the functions $\frac{1}{x}, \log(x), x\log(x)$ is convex and which is concave ?

Solution Recall Jensen inequality: if $E_P(|X|) < \infty$ and $f(x)$ is convex,

$$E_P(f(X)) \geq f(E_P(X))$$

If $f(x)$ is concave on the support of the probability distribution of X, then $x \mapsto -f(x)$ is convex, and we get

$$-E_P(f(X)) \geq -f(E_P(X)), \quad \text{which means } E_P(f(X)) \leq f(E_P(X)).$$

A twice differentiable function $f : \mathbb{R} \to \mathbb{R}$ is convex if and only if $f''(x) \geq 0 \; \forall x$ and it is concave if and only if $f''(x) \leq 0 \; \forall x$.

Now we see that $f(x) = 1/x$ is convex on $[0, \infty)$ since $f''(x) = 2x^{-3} \geq 0$, $\log(x)$ is concave on $[0, \infty)$ since $f''(x) = -x^{-2} \leq 0$ and $f(x) = x \log(x)$ is convex on $[0, \infty)$ since $f''(x) = x^{-1} \geq 0$.

3. Consider a sequence of independent and identically distributed random variables $(X_n(\omega) : n \in \mathbb{N})$, satisfying $X_n(\omega) \geq 0$ P-almosty surely and $E_P(X_1) = 1$. We also assume that $P(X_n = 1) < 1$ strictly. In such case necessarily $P(X_n > 1) > 0$ and $P(X_n < 1) > 0$.

   Let $Z_n(\omega) = \prod_{i=1}^{n} X_i(\omega)$.

   (a) Show that $Z_n \in L^1(P)$ and $E_P(Z_n) = 1$.
   Solution By P-independence

   $$E_P(Z_n) = E_P(X_1 X_2 \ldots X_n) = E_P(X_1) E_P(X_2) \ldots E_P(X_n) = 1^n = 1$$

   (b) Show that $\sqrt{Z_n} \in L^1(P)$ and $E_P(\sqrt{Z_n}) < 1$ strictly.
   Solution The function $f(x) = \sqrt{x}$ is strictly concave on $(0, \infty)$, Since $f''(x) = -x^{-3/2}/4 < 0$, the Jensen inequality (for concave function) holds strictly

   $$E_P(\sqrt{X_n}) < \sqrt{E_P(X_n)} = \sqrt{1} = 1$$

   and

   $$E_P(\sqrt{Z_n}) = E_P(\sqrt{X_1 X_2 \ldots X_n}) =$$
   $$E_P(\sqrt{X_1}) E_P(\sqrt{X_2}) \ldots E_P(\sqrt{X_n}) = E_P(\sqrt{X_1})^n \to 0$$

   as $n \to \infty$, since $E_P(\sqrt{X_1}) < 1$.

   (c) Use Chebychev inequality to show that $Z_n \xrightarrow{P} 0$ in probability.
   Hint: this is equivalent to show that $\sqrt{Z_n} \xrightarrow{P} 0$ in probability.
   Solution $\forall \eta > 0$

   $$P(Z_n > \eta) = P(\sqrt{Z_n} > \sqrt{\eta}) \leq \frac{E_P(\sqrt{Z_n})}{\sqrt{\eta}} = \frac{E_P(\sqrt{X_1})^n}{\sqrt{\eta}} \to 0$$

   as $n \to \infty$. By definition, it means that $Z_n \xrightarrow{P} 0$ in probability.

(d) Show that $\lim_{n\to\infty} Z_n(\omega) = 0$ P-almost surely.
Hint: Use Chebychev inequality and Borel Cantelli lemma to show that for any fixed $\eta > 0$,

$$P\big(Z_n \geq \eta \text{ infinitely often }\big) = 0,$$

$$\text{equivalently, } P\big(\sqrt{Z_n} \geq \sqrt{\eta} \text{ infinitely often }\big) = 0$$

Solution For any $\eta > 0$, let $A_n = \big\{\omega : \sqrt{Z_n} > \sqrt{\eta}\big\}$.

$$\sum_{n\in\mathbb{N}} P(A_n) \leq \eta^{-1/2} \sum_{n=0}^{\infty} E_P(\sqrt{X_1})^n = \frac{1}{(1 - E_P(\sqrt{X_1}))\sqrt{\eta}} < \infty$$

and the Borel Cantelli lemma implies that

$$P\big(\limsup A_n\big) = P\big(Z_n > \eta \text{ infinitely often }\big) = 0,$$

Therefore since countable union of P-null events has zero probability, by the definition of $\limsup$

$$P\Big(\bigcup_{m\in\mathbb{N}} \bigcap_{k\in\mathbb{N}} \bigcup_{n\geq k} \{\ Z_n > 1/m\}\Big) = 0,$$

and for the complement evant

$$P\Big(\bigcap_{m\in\mathbb{N}} \bigcup_{k\in\mathbb{N}} \bigcap_{n\geq k} \{\ Z_n \leq 1/m\}\Big) = 1,$$

which means that $\lim_{n\to\infty} Z_n(\omega) = 0$ P-almost surely.

(e) Write the definition of uniform integrability for a sequence of random variables and show that the sequence $(Z_n : n \in \mathbb{N})$ is not uniformly integrable.
Solution By definition a sequence $(X_n(\omega) : n \in \mathbb{N}) \subset L^1(P)$ is uniformly integrable if and only if

$$\lim_{K\to\infty} \sup_{n\in\mathbb{N}} E_P\big(|X_n| \mathbf{1}(|X_n|\ > K)\big) = 0.$$

Note that we have $E_P(Z_n) \equiv 1$ $\forall n \in \mathbb{N}$, but $Z_n(\omega) \to 0$ P-almost surely. This is in contradiction with uniform integrability, since for an uniformly integrable sequence X_n with $X_n(\omega) \to X(\omega)$ P-almost surely we always have

$$\lim_{n\to\infty} E_P(X_n) = E_P(\lim_n X_n) = E_P(X)\ .$$

4. Let $(G_i(\omega) : i \in \mathbb{N})$ be a sequence of independent and identically distributed standard Gaussian random variables, with $E_P(G_i) = 0$ and $E_P(G_n^2) = 1$, with moment generating function $E_P\big(\exp(tG_i)\big) = \exp(t^2/2)$, for $t \in \mathbb{R}$.

   Let

$$Z_n(\omega, t) = \exp\left(\theta \sum_{i=1}^{n} G_i(\omega)\right),$$

   and let $\tau(\omega)$ be an independent Poisson(λ) random variable, with parameter $\lambda > 0$, such that

$$P(\tau = k) = \exp(-\lambda)\frac{\lambda^k}{k}, \quad k \in \mathbb{N}$$

   Recall that the moment generating function of the Poisson(λ) distribution is given by $E_P(\exp(r\tau)) = \exp\big(\lambda\big(e^r - 1\big)\big)$ for $r \in \mathbb{R}$.

   Consider the random variable

$$Z_\tau(\omega) = \exp\left(t \sum_{i=1}^{\tau(\omega)} G_i(\omega)\right)$$

   (a) Write the definition of conditional expectation.

   Solution Check the lecture notes or a book.

   (b) Compute the conditional expectation

$$E_P(Z_\tau | \ \sigma(G_i : i \in \mathbb{N}))(\omega)$$

   Solution Since τ and $(G_i : i \in \mathbb{N})$ are P-independent, we compute this conditional expectation by fixing the values of the G_i and integrating out τ under the Poisson(λ) measure.

$$E_P(Z_\tau | \ \sigma(G_i : i \in \mathbb{N}))(\omega) = e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \exp\left(t \sum_{i=1}^{k} G_i(\omega)\right)$$

   (c) For n fixed, compute the conditional expectation

$$E(Z_\tau | \ \sigma(G_i : i \leq n))(\omega)$$

   SolutionSince τ and $(G_i : i \in \mathbb{N})$ are P-independent, and the G_i variables are mutually independent we compute this conditional expectation by fixing the values of the G_i for $i = 1, \dots, n$ and integrating out τ and $(G_i : i > n)$ under the product of standard Gaussian probability measures:

$$E(Z_\tau | \, \sigma(G_i : i \le n))(\omega) = e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \exp\Bigl(t \sum_{i=1}^{n\wedge k} G_i(\omega) \Bigr) E_P(\exp(tG_i))^{k-(n\wedge k)}$$

$$= e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \exp\Bigl(t \sum_{i=1}^{n\wedge k} G_i(\omega) \Bigr) \exp\bigl(t^2 \bigl(k - (n \wedge k)\bigr)/2\bigr)$$

(d) Compute the conditional expectation

$$E(Z_\tau | \sigma(\tau))(\omega)$$

Solution Since τ and $(G_i : i \in \mathbb{N})$ are P-independent, and the G_i variables are mutually independent we compute this conditional expectation by fixing the values of $\tau(\omega)$ and integrating out τ and $(G_i : i \in \mathbb{N})$ under the product of standard Gaussian probability measures.

$$E(Z_\tau | \sigma(\tau))(\omega) = E_P\Biggl(\exp\Bigl(t \sum_{i=1}^{k} G_i(\omega) \Bigr) \Biggr) \Bigg|_{k=\tau(\omega)} = \exp\bigl(\tau(\omega) t^2/2\bigr)$$

(e) Compute the expectation $E(Z_\tau)$. **Solution** Since the expectation of the conditional expectation is the expectation of the random variable,

$$E_P(Z_\tau) = E_P\Bigl(E(Z_\tau | \sigma(\tau)) \Bigr) = E_P\Bigl(\exp(\tau\theta) \Bigr)$$

where $\theta = t^2/2$, which gives

$$E_P(Z_\tau) = \exp\Bigl(\lambda(e^{t^2/2} - 1) \Bigr)$$

Hint: remember that if X is P-independent from the sub σ-algebra $\mathcal{G}$ and Y is a $\mathcal{G}$-measurable random variable, for every non-negative measurable test function $f(x, y)$ we have

$$E_P\bigl(f(X, Y) | \mathcal{G}\bigr)(\omega) = E_P\bigl(f(X, y)\bigr) \Bigg|_{y=Y(\omega)}$$