UH Probability Theory II, Fall 2015, Exercises 12 (9.12.2015)

In all problems the random variables live in the probability space $(\Omega, \mathcal{F}, P)$ and $\mathcal{G} \subseteq \mathcal{F}$ is a sub σ-algebra of $\mathcal{F}$.

1. Consider random variables $X(\omega), Y(\omega), Z(\omega) \in L^2(\Omega, \mathcal{F}, P)$. Let

   $$H = \Big\{ \ a(Y) + b(Y)Z : a, b \text{ Borel mitalliset funktiot} \Big\} \cap L_2(\Omega, \sigma(Y, Z), P)$$

   H contains the square-integrable and $\sigma(Z, Y)$ measurable random variables which are linear functions of $Z(\omega)$ with $\sigma(Y)$-measurable coefficients.

   - Show that

     $$\hat{X}(\omega) = E_P(X|\sigma(Y)) + \Big(Z(\omega) - E_P(Z|\sigma(Y))(\omega) \Big) \frac{\text{Cov}_P(XZ| \ \sigma(Y))(\omega)}{\text{Var}_P(Z^2| \ \sigma(Y))(\omega)}$$

     is the orthogonal projection of a random variable $X \in L^2(\Omega, \mathcal{F}, P)$ into the subspace H.
   - Compute the conditional mean square error $E_P((\hat{X} - X)^2 | \sigma(Z, Y))(\omega)$.
   - Compute the conditional mean square error $E_P((\hat{X} - X)^2 | \sigma(Y))(\omega)$.
   - Compute the conditional mean square error $E_P((\hat{X} - X)^2)$.
   - Let $V(\omega) = E_P(X|\sigma(Z, Y))(\omega)$. Show that

     $$E_P((V - X)^2 | \sigma(Y, Z))(\omega) \leq E_P((\hat{X} - X)^2 | \sigma(Y, Z))(\omega)$$
     $$E_P((V - X)^2 | \sigma(Y))(\omega) \leq E_P((\hat{X} - X)^2 | \sigma(Y))(\omega)$$
     $$E_P((V - X)^2) \leq E_P((\hat{X} - X)^2)$$

2. On a probability space $(\Omega, \mathcal{F}, P)$, let $N(\omega)$ be a random variable with geometric distribution

   such that $P(N = k) = (1-p)^{k-1} p$ kun $k \in \mathbb{N}$, with parameter $p \in (0, 1)$,

   and let $\{X_k(\omega) : \ k \in \mathbb{N}\}$ a sequence of P-independent and identically distributed standard Gaussian random variables.

   Recall that the σ-algebra generated by the random variables $\{X_j(\omega)\}_{j \in \mathcal{J}}$ denoted by

   $$\sigma\big(X_j(\omega) : j \in \mathcal{J}\big)$$

is the smallest σ-algebra containing the events

$$\{\omega : (X_{j_1}(\omega), X_{j_2}(\omega), \dots, X_{j_n}(\omega)) \in B_n\}$$

where $n \in \mathbb{N}$, $\{j_1, \dots j_n\} \subseteq \mathcal{J}$ is a finite subset of indexes and $B_n \in \mathcal{B}(\mathbb{R}^n)$ is a Borel set.

Let

$$Y(\omega) := \sum_{k=1}^{N(\omega)} X_k(\omega) = \sum_{k=1}^{\infty} X_k(\omega) \mathbf{1}(k \le N(\omega))$$

(a) Show that$Y(\omega)$ is a random variable.

(b) Conmpute the conditional expectations

$$E_P(Y|\sigma(N))(\omega) \quad \text{ja} \quad E_P(Y|\sigma(X_k : k \in \mathbb{N}))(\omega),$$

and show that they are square integrable under P.

(c) Compute the conditional expectatations

$$E_P(Y^2|\sigma(N))(\omega) \quad \text{ja} \quad E_P(Y^2|\sigma(X_k : k \in \mathbb{N}))(\omega),$$

and show that they are integrable w.r.t. P.

(d) Compute the expectations $E_P(Y)$, $E_P(Y^2)$.

3. We construct Markov chain $(X_t(\omega) : t \in \mathbb{N})$ taking values in $\mathbb{R}^d$ with initial probability distribution $P(X_0 \in B) = \pi(B)$ and transition kernels $K_t(B, x)$, where $B \in \mathcal{B}(\mathbb{R}^d)$, $x \in \mathbb{R}^d$ and $t \in \mathbb{N}$, meaning that the map $B \mapsto K_t(B, x)$ is a probability measure for each x, t fixed, and the map $x \mapsto K_t(B, x)$ is Borel-measurable for each B, t fixed.

   It means that $\forall t$, $B_0, B_1, \dots, B_t \in \mathcal{B}(\mathbb{R}^d)$

$$P(X_0 \in B_0, X_1 \in B_1, \dots, X_t \in B_t) =$$
$$\int_{B_0} \int_{B_1} \int_{B_2} \dots \int_{B_t} K_t(dx_t, x_{t-1}) \dots K_2(dx_2, x_1) K_1(dx_1, x_0) \pi(dx_0)$$

   (a) Check that Kolmogorov extension theorem implies the existence of such stochastic process.

(b) Define the operators K_t, $t \in \mathbb{N}$ operating on measurable functions with

$$(K_t f)(x) = \int_{\mathbb{R}^d} f(y) K_t(y, dx) := E_x(f(X_1))$$

Show that when $f(x)$ is bounded and measurable, the stochastic process defined by $M_0(f) = 0$, and

$$M_t(f) = \sum_{s=1}^{t} \big(f(X_s) - (K_s f)(X_{s-1}) \big)$$

is a P-martingale in the filtration $\mathbb{F} = (\mathcal{F}_t : t \in \mathbb{N}\})$ where $\mathcal{F}_t = \sigma(X_s : s = 0, 1, \ldots, t)$.

(c) Compute the Doob martingale decomposition

$$f(X_t) = f(X_0) + M_t(f) + A_t(f)$$

where $A_t(f)$ is predictable w.r.t. the filtration $\mathbb{F}$, meaning that $\forall t \in \mathbb{N}$, $A_t(f)$ is $\mathcal{F}_{t-1}$-measurable.

(d) Let $(G_t : t \in \mathbb{N})$ independent and identically distributed Gaussian random variables, and let $X_0 = 0$, $X_t = G_1^2 + G_2^2 + \cdots + G_t^2$. Find the transition kernels of X_t as the regular transition version of the conditional probabilities

$$P(X_t \in B | \sigma(X_{t-1})(\omega) = K_t(X_{t-1}(\omega), B)$$

(e) Compute the Doob martingale decomposition of (X_t) in the filtration $\mathbb{F} = (\mathcal{F}_t : t \in \mathbb{N}\})$ where $\mathcal{F}_t = \sigma(X_s : s = 0, 1, \ldots, t)$.

(f) Compute the Doob martingale decomposition of $Y_t = \sqrt{X_t}$ in the filtration $\mathbb{F} = (\mathcal{F}_t : t \in \mathbb{N}\})$ where $\mathcal{F}_t = \sigma(X_s : s = 0, 1, \ldots, t)$.

(g) Compute the Doob martingale decomposition of X_t and $Y_t = \sqrt{X_t}$ also the larger filtration $\widetilde{\mathbb{F}} = (\widetilde{\mathcal{F}_t} : t \in \mathbb{N}\})$ where $\widetilde{\mathcal{F}}_t = \sigma(G_s : s = 0, 1, \ldots, t) \supset \sigma(X_s : s = 0, 1, \ldots, t)$.

4. Prove Fatou lemma for the conditional exprctation: if $0 \le X_n(\omega)$, $\forall n \in \mathbb{N}$ P-almost surely

$$0 \le E_P(\liminf X_n | \mathcal{G})(\omega) = \liminf_n E_P(X_n | \mathcal{G})(\omega)$$

Hint: use the monotone convergence theorem for the conditional expectation.

5. Let $X(\omega)$ and $Y(\omega)$ P-independent and uniformly distributed on $[0,1]$, meaning that

$$P(X \in dx, Y \in dy) = \mathbf{1}_{[0,1]}(x)\ \mathbf{1}_{[0,1]}(y)\ dx\ dy$$

Let $Z(\omega) = \min(X(\omega), Y(\omega))$ and $I(\omega) = \mathbf{1}(X(\omega) \le Y(\omega))$

(a) Compute $P(X > Y)$.

(b) Compute the "elementary" conditional expectation conditioned to a P-positive event

$$E_P(X|X > Y)$$

(c) Compute the conditional expectation

$$E_P(X|\sigma(I))(\omega)$$

(d) Compute the conditional expectation $E_P(X|\sigma(Z), I)(\omega)$.
Hint Since

$$Z(\omega)\mathbf{1}(X(\omega) > Y(\omega)) = Y(\omega)\mathbf{1}(X(\omega) > Y(\omega))$$

you can check by the Kolmogorov definition of conditional expectation that

$$\begin{aligned}
&E_P(X|\sigma(Z), \sigma(I))(\omega)\ \mathbf{1}(X(\omega) > Y(\omega))\\
&= E_P(X|\sigma(Y), \sigma(I))(\omega)\mathbf{1}(X(\omega) > Y(\omega))\\
&= \frac{E_P(X\mathbf{1}(X > Y)|\sigma(Y))}{P(X > Y|\sigma(Y))(\omega)}\mathbf{1}(X(\omega) > Y(\omega))
\end{aligned}$$

Recall that for P-independent random variables X and Y

$$E_P(f(X,Y)|\sigma(Y))(\omega) = E_P(f(X,y))\Big|_{y=Y(\omega)}$$

Hint compute first $E_P(X|\sigma(Z,I))$, with $I(\omega) := \mathbf{1}(X(\omega) \le Y(\omega))$ knowing that $\sigma(Z,I) \supseteq \sigma(Z)$.

$$E_P(Z|\sigma(Z)) = E_P\big(E_P(X|\sigma(Z,I))\big|\sigma(Z)\big)$$