HU, Probability Theory Fall 2015, Problems 7 (28.10.2015)
In the problems all random variables live in a probability space $(\Omega, \mathcal{F}, P)$.

1. On a probability space $(\Omega, \mathcal{F}, P)$,

   let $(X_n(\omega) : n \in \mathbb{N})$ be a sequence of exponential random variables such that

   $$P\big(X_1 > t_1, \dots, X_n > t_n\big) = \exp\left(-\lambda \sum_{i=1}^{n} t_i\right) \qquad \forall n \in \mathbb{N}, t_1, \dots, t_n \geq 0,$$

   where $\lambda > 0$ is a parameter.

   (a) Show that the random variables $(X_n(\omega) : n \in \mathbb{N})$ are independent under P.

   Solution

   $$P\big(X_1 > t_1, \dots, X_n > t_n\big) = \exp\left(-\lambda \sum_{i=1}^{n} t_i\right) = \prod_{i=1}^{n} \exp\Big(-\lambda t_i\Big) = \prod_{i=1}^{n} P\big(X_i > t_i\big)$$
   $$\forall n \in \mathbb{N}, t_1, \dots, t_n \geq 0,$$

   and since the collection of rectangles

   $$\mathcal{I} = \Big\{(t_1, 1] \times (t_2, 1] \times \cdots \times (t_n, 1] : t_i \in [0, 1]\Big\}$$

   form a π-system (closed under intersections) which generate the borel σ-algebra $\mathcal{B}([0,1]^n) = \mathcal{B}([0,1])^{\otimes n}$, by Dynkin lemma on uniqueness of probability measures,

   $$P\big(X_1 \in B_1, \dots, X_n \in B_n\big) = \prod_{i=1}^{n} P\big(X_i \in B_i\big) \qquad \forall B_i \in \mathcal{B}([0,1]), 1 \leq i \leq n$$

   which means that $X_1, \dots, X_n$ are P-independent, and they are also identically distributed with λ-exponential distribution.

   (b) Let

   $$Y_n(\omega) := \min\big\{X_1(\omega), X_2(\omega), \dots, X_n(\omega)\big\} .$$

   Compute $P(Y_n > t)$, and compute also the probability density function of Y_n .

Solution

$$P(Y_n > t) = P(X_1 > t, \dots, X_n > t) = \prod_{i=1}^{n} P(X_i > t) = \exp(-\lambda t)^n = \exp(-n\lambda t)$$

and the probability density function of Y_n is given by

$$f_{Y_n}(t) := -\frac{d}{dt}P(Y_n > t) = -\frac{d}{dt}\exp(-n\lambda t) = n\lambda\exp(-n\lambda t)$$

which is the density of a λ-exponential random variable with parameter $n\lambda$.

(c) Let $X_n^*(\omega) = \max\bigl\{X_1(\omega), X_2(\omega), \dots, X_n(\omega)\bigr\}$
Compute $P(X_n^* \le t)$. Compute also the probability density function of X_n^*. **Solution**

$$P(X_n^* \le t) = P(X_1 \le t, X_2 \le t, X_n \le t) = \prod_{i=1}^{n} P(X_i \le t) = \bigl(1 - \exp(-\lambda t)\bigr)^n$$

and the probability density function of X_n^* is given by

$$f_{X_n^*}(t) := \frac{d}{dt}P(X_n^* \le t) = n\lambda\bigl(1 - \exp(-\lambda t)\bigr)^{n-1}\exp(-\lambda t)$$

(d) Compute $\lim_{n\to\infty} P\Bigl(\lambda X_n^* \le t + \log(n)\Bigr)$.
Hint: $\bigl(1 + x/n\bigr)^n \longrightarrow \exp(x)$ as $n \to \infty$.
Solution

$$P\Bigl(\lambda X_n^* \le t + \log(n)\Bigr) = \bigl(1 - \exp(-t - \log(n))\bigr)^n = \Bigl(1 - \frac{\exp(-t)}{n}\Bigr)^n \tag{0.1}$$

$$\longrightarrow \exp\bigl(-\exp(-t)\bigr)$$

as $n \to \infty$. The function

$$G(t) = \exp\bigl(-\exp(-t)\bigr)$$

is continuous and strictly increasing, with $G(-\infty) = 0$ and $G(+\infty) = 1$, it is the cumulative distribution function of a probability distribution on $\mathbb{R}$. Such distribution is called Gumbel extreme value distribution, which in the context of extreme value theory.

Let $\xi_n(\omega) = (\lambda X^* n(\omega) - \log n)$. We have shown that $G_n(t) := P(\xi_n \le t) \to G(t)$. When a sequence of cumulative distribution functions $G_n(t)$ converges to a cumulative distribution function $G(t)$ at all points t where $G(t)$ is continuous, we say that ξ_n converges in law to the limiting distribution G.

2. Consider a sequence of random variables $(U_k(\omega) : k \in \mathbb{N})$ such that for $\forall t_1, \dots, t_n \in [0,1]$,

$$P(U_1 \le t_1, \dots, U_n \le t_n) = \prod_{k=1}^{n} t_k$$

(a) Show that $(U_k(\omega) : k \in \mathbb{N})$ are independent and uniformly distributed on $[0,1]$.

Solution

$$P(U_1 \le t_1, \dots, U_n \le t_n) = \prod_{k=1}^{n} t_k = \prod_{k=1}^{n} P(U_k \le t_k)$$

which implies as in the previous exercise that $U_1, \dots, U_n$ are P-independent, and also that they are identically distributed where the common probability distribution is Lebesgue measure on the interval $[0,1]$.

(b) Consider $\overline{U}_n(\omega) = \max\{U_1(\omega), \dots, U_n(\omega)\}$.

Compute the cumulative distribution function of $\overline{U}_n$, $F_{\overline{U}_n}(t) = P(\overline{U}_n \le t)$.

Solution

$$P(\overline{U}_n \le t) = P(U_1 \le t, \dots, U_n \le t) = \prod_{i=1}^{n} P(U_i \le t) = t^n$$

(c) Show that $\lim_{n\to\infty} \overline{U}_n(\omega) = 1$ $\mathbb{P}$-almost surely. **Solution** Note that $\forall \omega \in \Omega$,

$$0 \le \overline{U}_n(\omega) \le \overline{U}_{n+1}(\omega) \uparrow \overline{U}_\infty(\omega) \le 1$$

where the limit exists since the sequence $\overline{U}_n(\omega)$ is non-decreasing and bounded by 1 for each ω.

We show that $P(\overline{U}_\infty = 1) = 1$.

Equivalently we will show that

$$0 \stackrel{?}{=} P\big(\overline{U}_\infty < 1\big) = P\Big(\bigcup_{m\in\mathbb{N}} \{\overline{U}_\infty < 1 - 1/m\}\Big)$$

and for each fixed $m \in \mathbb{N}$

$$\{\overline{U}_\infty < 1 - 1/m\} = \bigcap_{n\in\mathbb{N}} \{U_n < 1 - 1/m\}$$

But by the σ-additivity of P,

$$P\Big(\bigcap_{n\in\mathbb{N}}\{U_n < 1 - 1/m\}\Big) = \lim_{N\to\infty} P\Big(\bigcap_{1\le n\le N}\{U_n < 1 - 1/m\}\Big) = \lim_{N\to\infty}\big(1 - 1/m\big)^N = 0$$

and the claim follows since the countable union of P-null events is a P-null event.

(d) Let $\underline{U}_n(\omega) = \min\big\{U_1(\omega), \ldots, U_n(\omega)\big\}$.
Compute the cumulative distribution function of $\underline{U}_n$, $F_{\underline{U}_n}(t) = P(\underline{U}_n \le t)$.
Solution

$$P(\underline{U}_n \le t) = 1 - P(\underline{U}_n > t) = 1 - P(U_1 > t, \ldots, U_n > t) =$$
$$1 - \prod_{i=1}^{n} P(U_i > t) = 1 - (1-t)^n$$

(e) Show that $\lim_{n\to\infty} \underline{U}_n(\omega) = 0$ $\mathbb{P}$-almost surely.
Hint: $V_n = (1-U_n)$ has the same distribution as U_n, which implies that $\underline{U}_n$ and $(1-\overline{U}_n)$ have the same distribution.
Solution
Note that $\forall\omega \in \Omega$,

$$1 \ge \underline{U}_n(\omega) \ge \underline{U}_{n+1}(\omega) \downarrow \underline{U}_\infty(\omega) \ge 0$$

where the limit exists since the sequence $\overline{U}_n(\omega)$ is non-increasing and non-negative for each ω.
We show that $P\big(\underline{U}_\infty = 0\big) = 1$.
Equivalently we will show that

$$0 \stackrel{?}{=} P\big(\underline{U}_\infty > 0\big) = P\Big(\bigcup_{m\in\mathbb{N}}\{\underline{U}_\infty > 1/m\}\Big)$$

and for each fixed $m \in \mathbb{N}$

$$\{\underline{U}_\infty > 1/m\} = \bigcap_{n\in\mathbb{N}} \{U_n > 1/m\}$$

But by the σ-additivity of P,

$$P\biggl(\bigcap_{n\in\mathbb{N}} \{U_n > 1/m\}\biggr) = \lim_{N\to\infty} P\biggl(\bigcap_{1\le n\le N} \{U_n > 1/m\}\biggr) = \lim_{N\to\infty} (1-1/m)^N = 0$$

and the claim follows since the countable union of P-null events is a P-null event.

3. (a) let $X(\omega), X_n(\omega), n \in \mathbb{N}$ such that $X_n(\omega) \to X(\omega)$ P-almost surely. Show that also the Cesaro mean converges P-almost surely to X

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^n X_i(\omega) = X(\omega) \qquad P\text{-almost surely}$$

Solutions Byt assumption, there exists an N with $P(N) = 0$ such that $\forall \omega \in N^c, \forall \varepsilon > 0\ \exists K(\varepsilon,\omega)$ such that $\forall n \ge K(\varepsilon,\omega)$,

$$| X_n(\omega) - X(\omega)| < \varepsilon$$

Let

$$\bar{S}_n(\omega) = \frac{1}{n} \sum_{i=1}^n X_i(\omega)$$

and

$$\bar{S}_n(\omega) - X(\omega) = \frac{1}{n} \sum_{i=1}^n \bigl(X_i(\omega) - X(\omega)\bigr).$$

By the triangle inequality, when $n \ge K(\varepsilon,\omega)$

$$\bigl|\bar{S}_n(\omega) - X(\omega)\bigr| \le \frac{1}{n} \sum_{i=1}^{K(\varepsilon,\omega)-1} \bigl|X_i(\omega) - X(\omega)\bigr| + \frac{1}{n} \sum_{i=K(\varepsilon,\omega)}^{n} \bigl|X_i(\omega) - X(\omega)\bigr| \tag{0.2}$$

$$\le \frac{1}{n} C(\omega,\varepsilon) + \varepsilon$$

where $C(\omega, \varepsilon)$ does not depend from n. Therefore when $\omega \in N^c$, $\forall \varepsilon > 0$

$$\lim_{n\to\infty} \left|\bar{S}_n(\omega) - X(\omega)\right| \le \varepsilon$$

and since $\varepsilon > 0$ is arbitrary this means that

$$\lim_{n\to\infty} \bar{S}_n(\omega) = X(\omega) \quad \forall \omega \in N^c.$$

(b) Assume now that $E_P\big(|X_n - X|\big) \to 0$, as $n \to \infty$ (without assuming P-almost sure convergence). We also need to assume that $E_P(|X_n|) < \infty$ $\forall n$, in order to guarantee that the Cesaro means $\bar{S}_n$ are integrable.
Show that the Cesaro mean is converging in $L^1(P)$, that is

$$\lim_{n\to\infty} E_P\left(\left|\left\{ \frac{1}{n}\sum_{i=1}^n X_i \right\} - X(\omega)\right|\right) \to 0 \qquad \text{as } n \to \infty$$

Hint: note that by the triangle inequality

$$\left|\left\{ \frac{1}{n}\sum_{i=1}^n X_i \right\} - X(\omega)\right| \le \frac{1}{n}\sum_{i=1}^n \left|X_i - X(\omega)\right| =$$
$$\frac{1}{n}\sum_{i=1}^M \left|X_i - X(\omega)\right| + \frac{1}{n}\sum_{j=M+1}^n \left|X_j - X(\omega)\right|$$

$\forall n \ge M$, where the inequalities are preserved after taking the expectation.

Solution By assumption $\forall \varepsilon > 0$ $\exists K(\varepsilon)$ (which now does not depend on ω) such that $\forall n \ge K(\varepsilon)$,

$$E\big(| X_n - X|\big) < \varepsilon$$

By the triangle inequality

$$\left|\bar{S}_n(\omega) - X(\omega)\right| \le \frac{1}{n}\sum_{i=1}^{K(\varepsilon)-1} \left|X_i(\omega) - X(\omega)\right| + \frac{1}{n}\sum_{i=K(\varepsilon)}^{n} \left|X_i(\omega) - X(\omega)\right| \le \frac{1}{n}C(\omega,\varepsilon) + \varepsilon \tag{0.3}$$

where

$$C(\omega, \varepsilon) = \sum_{i=1}^{K(\varepsilon)-1} \left|X_i(\omega) - X(\omega)\right|$$

In order to make sure that $C(\omega, \varepsilon)$ is integrable, we assume that $E_P(|X_n|) < \infty \ \forall n$. This means that $|X| \leq |X_n| + |X - X_n|$ is integrable since $E(|X|) \leq E(|X_n|) + E(|X - X_n|) < \infty$ for n large enough, and since X and X_n are integrable $\forall n$ also $|X - X_n| \leq |X| + |X_n|$ is integrable $\forall n$. Therefore $C(\omega, \varepsilon)$ in (0.3) is integrable and does not depend on n. By taking expectation in (0.3)

$$E\big(\big|\bar{S}_n - X\big|\big) \leq \frac{1}{n} E(C(\varepsilon)) + \varepsilon$$

with $E(C(\varepsilon)) < \infty$, and

$$\lim_{n \to \infty} E\big(\big|\bar{S}_n - X\big|\big) \leq \varepsilon, \quad \forall \varepsilon > 0$$

which implies that $\bar{S}_n \stackrel{L^1(P)}{\to} X$ in $L^1(P)$-norm.

4. Let $X(\omega), (X_n(\omega) : n \in \mathbb{N})$, random variables on a probability space $(\Omega, \mathcal{F}, P)$.

   Show that if $\forall \varepsilon > 0$

   $$\sum_{n=0}^{\infty} P(|X_n(\omega) - X(\omega)| > \varepsilon) < \infty$$

   it follows $\lim_{n \uparrow \infty} X_n(\omega) = X(\omega)$ P-almost surely.

   Hint: show first that

   $$\big\{\omega : X_n(\omega) \not\to X(\omega)\big\} = \bigcup_{k \in \mathbb{N}} \{\omega : |X_n(\omega) - X(\omega)| > k^{-1} \text{ infinitely often } \}$$

   and recall Borel-Cantelli's lemma.

   Solution Let

   $$A_n^m = \big\{\omega : |X_n(\omega) - X(\omega)| > 1/m\big\}$$

   Since by assumption $\forall m \in \mathbb{N}$

   $$\sum_{n \to \infty} P(A_n^m) < \infty$$

   by the first Borel Cantelli lemma $\forall m \in \mathbb{N}$

   $$P(\limsup_{n \in \mathbb{N}} A_n^m) = 0$$

and since the countable union of P-null events is a P-null event

$$0 = P\Big(\bigcup_m \bigcap_k \bigcup_{n \geq k} A_n^m \Big)$$

For the complement of this event we have

$$\begin{aligned} 1 &= P\Big(\bigcap_{m\in\mathbb{N}} \bigcup_{k\in\mathbb{N}} \bigcap_{n\geq k} (A_n^m)^c \Big) \\ &= P\Big(\Big\{ \omega : \forall m \exists K = K(\omega, m) \text{ such that } \forall n \geq K(\omega, m)\ |X_n(\omega) - X(\omega)| \leq 1/m \Big\} \Big) \\ &= P\Big(\Big\{ \omega : \lim_{n\to\infty} X_n(\omega) = X(\omega) \Big\} \Big) \end{aligned}$$

5. Consider a random variable $X(\omega)$ with $E_P(|X|) < \infty$. Show that

$$E_P\big(|X|\mathbf{1}(|X| > n)\big) = \int_\Omega |X(\omega)|\mathbf{1}\big(|X(\omega)| > n\big)P(d\omega) \to 0 \text{ as } n \to \infty \ .$$

Note that $\forall \omega \in \Omega$,

$$\lim_{n\to\infty} |X(\omega)|\mathbf{1}\big(|X(\omega)| > n\big) = 0$$

simply because $X(\omega) \in R$, and $\mathbf{1}\big(|X(\omega)| > n\big) = 0$ for all $n \geq |X(\omega)|$. Note also that $\forall \omega \in \Omega$

$$0 \leq |X(\omega)|\mathbf{1}(|X(\omega)| > n) \leq |X(\omega)|$$

where the upper bound is integrable by the assumption $E(|X|) < \infty$. Therefore Lebesgue dominated convergence Theorem applies and we can change the order of limit and expectation obtaining

$$\lim_{n\to\infty} E_P\big(|X|\mathbf{1}(|X| > n)\big) = E_P\Big(|X| \lim_{n\to\infty} \mathbf{1}(|X| > n) \Big) = 0$$