HU, Probability Theory Fall 2015, Solutions to Problems 5 (7.10.2015)

1. Consider the probability space $\Omega = [0,1]$ equipped with the Borel σ-algebra $\mathcal{F} = \mathcal{B}([0,1])$ and the uniform probability measure $\mathbb{P}$ such that $\mathbb{P}((a,b]) = b - a$ for $0 \le a \le b \le 1$, which is also called Lebesgue measure.

   Show that the identity map $U : \Omega \to [0,1]$ with $U(\omega) = \omega$ is an uniformly distributed random variable, which means

   $\mathbb{P}(\{ \omega : U(\omega) \in (a,b]\}) = b - a.$

   Solution This is trivial, since $\omega \to U(\omega)$ is the identity map on $[0,1]$:

   $$\mathbb{P}(\{ \omega : U(\omega) \in (a,b]\}) = \mathbb{P}(\omega : \omega \in (a,b]) = \mathbb{P}((a,b]) = b - a$$

   when $0 \le a \le b \le 1$.

   Let now $(\Omega, \mathcal{F}, \mathbb{P})$ be an abstract probability space and $U : \Omega \to [0,1]$ a random variable with uniform distribution on $[0,1]$, which means $\mathbb{P}(\{ \omega : U(\omega) \in (a,b]\}) = b - a$.

   Let $F : \mathbb{R} \to [0,1]$ a cumulative probability distribution function (c.d.f.), which is right continuous, non-decreasing with $F(+\infty) = 1$ and $F(-\infty) = 0$.

   We shall construct a random variable on $(\Omega, \mathcal{F})$ with values in $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ such that

   $$\mathbb{P}(\{\omega : \; X(\omega) \le t\}) = F(t)$$

   Assume for simplicity that $F(t)$ is continuous and stricty increasing, with $F(s) < F(t) \; \forall s < t$.

   In this case there is an unique inverse $F^{-1} : [0,1] \to \mathbb{R}$ such that $F(F^{-1}(u)) = u \; \forall u \in [0,1]$ and $F^{-1}(F(t)) = t \; \forall t \in \mathbb{R}$.

   Show that $X(\omega) = F^{-1}(U(\omega))$ is a random variable with

   $P(\{X(\omega) \le t\}) = F(t)$.

   Using a generalized inverse, this construction extends also to the general cumulative distribution function, which does not need to be continous from the left neither strictly increasing.

   Solution $\mathbb{P}(X(\omega) \le t) = \mathbb{P}(F^{-1}(U) \le t) = \mathbb{P}(F(F^{-1}(U) \le F(t)) = \mathbb{P}(U \le F(t)) = F(t)$. When F is not continuous strictly increasing, it can have jumps, which corresponds to points t with $\Delta F(t) = \mathbb{P}(\{X = t\}) > 0$, and also $\mathbb{P}(X \in (a,b]) = 0$ for intervals with $F(a) = F(b)$.

In such cases we define the generalized inverse as

$$F^{-1}(u) = \sup\{t : F(t) \le u\} = \inf\{t : F(t) > u\}$$

Note that when $\Delta F(t) > 0$ $F^{-1}(u) = t \iff u \in [F(t-), F(t)]$, and $\mathbb{P}(F^{-1}(U) = t) = \mathbb{P}(U \in [F(t-), F(t)]) = \Delta F(t)$.

Also when $F(a) = F(b)$, $\mathbb{P}(F^{-1}(U) \in (a,b]) = \mathbb{P}(U = F(b)) = 0$.

Then

$$\begin{aligned} &\mathbb{P}(u : F^{-1}(u) \le t) = \mathbb{P}(u : \sup\{s : F(s) \le u\} \le t) \\ &= \mathbb{P}(u : F(s) \le u \Longrightarrow s \le t) = \mathbb{P}(u : u \le F(t)) = F(t) \end{aligned}$$

This construction of a random variable with given cumulative distribution function on the probability space $\Omega = [0,1]$ equipped with the uniform probability is called *Skorokhod representation*. It can be used simultaneously for several distributions $F_1(t), \dots, F_n(t)$ to construct by using the same uniform random variable $U(\omega)$ a coupling $X_1(\omega) = F_1^{-1}(U(\omega)), \dots, X_n(\omega) = F_n^{-1}(U(\omega))\big)$, where the variables are dependent with given marginals distributions $F_i(t), i = 1, \dots, n$.

2. On an abstract probability space $(\Omega, \mathcal{F}, \mathbb{P})$, let $X(\omega) \ge 0$ $\forall \omega \in \Omega$ a non-negative random variable.

   We have defined the expectation of as

$$E_{\mathbb{P}}(X) = \sup_{0 \le Y \le X, \text{ with } Y \in \mathcal{S}F^+} E_{\mathbb{P}}(Y)$$

   where the supremum is taken over the simple random variables Y (taking finitely many values) such that $0 \le Y(\omega) \le X(\omega)$ $\forall \omega \in \Omega$

   Assume that $X(\omega) \in \mathbb{N}$ $\forall \omega \in \Omega$.

   (a) Show that

$$E_{\mathbb{P}}(X) = \sum_{n=1}^{\infty} n \mathbb{P}(\{\ \omega : X(\omega) = n\}) \ = \sum_{n=1}^{\infty} n P_X(\{n\})$$

   where $P_X(\{n\}) = \mathbb{P}(\{\omega : X(\omega) = n\})$ is the distribution of X with $E_{\mathbb{P}}(X) \in [0, +\infty]$ (the series may also diverge).

(b) Show one non-trivial example with $X(\omega)$ taking countably many values in $\mathbb{N}$ and choosing the distribution of X $P_X(\{n\})$ such that $E_{\mathbb{P}}(X) < \infty$, and another example where $E_{\mathbb{P}}(X) = +\infty$.

Solution Of course this follows by applying the monotone convergence Theorem, but we are asked to give a direct short proof without using the theorem. Let $0 \le Y^{(N)}(\omega) \le X(\omega)$ a sequence of simple random variables such that $E_{\mathbb{P}}(Y^{(N)}) \to E_{\mathbb{P}}(X)$. Note that this approximating sequence does not need to be monotone ! One can always obtain a monotone sequence by taking

$$0 \le Y^{(N)}(\omega) \le \hat{Y}^{(N)}(\omega) := \max_{k \le N} \hat{Y}^{(k)}(\omega) \le \hat{Y}^{(N+1)}(\omega) \le X(\omega)$$

and since the expectation of simple random variables is a positive operator

$$0 \le E_{\mathbb{P}}\big(Y^{(N)}\big) \le E_{\mathbb{P}}\big(\hat{Y}^{(N)}\big) \le E_{\mathbb{P}}\big(\hat{Y}^{(N+1)}\big) \le E_{\mathbb{P}}(X)$$

where $E_{\mathbb{P}}\big(Y^{(N)}\big) \to E_{\mathbb{P}}(X)$ implies that $E_{\mathbb{P}}\big(Y^{(N)}\big) \uparrow E_{\mathbb{P}}(X)$ monotonically.

By the definition of the expectation such sequence exists, since

$$E_{\mathbb{P}}(X) = \sup_{0 \le Y \le X} \big\{ E_{\mathbb{P}}(Y) \big\}$$

where the supremum is taken over the simple random variables. Since each $Y^{(N)}$ takes finitely many values, it is a bounded random variable, $0 \le Y^{(N)}(\omega) \le K_N \ \forall \omega \in \mathbb{N}$. By repeating elements of the sequence we can find another subsequence of simple random variables with $0 \le \widetilde{Y}^{(N)} \le X(\omega)$, $\widetilde{Y}^{(N)}(\omega) \le N \ \forall N \in \mathbb{N}$ and $E_{\mathbb{P}}(\widetilde{Y}^{(N)}) \to E_{\mathbb{P}}(X)$.

But then $Y^{(N)}(\omega) \le X(\omega) \wedge N$ where on the right hand side we have also a simple random variable. Now since the expectation is a positive operator on the space of simple random variables

$$E_{\mathbb{P}}(Y^{(N)}) \le E_{\mathbb{P}}(X \wedge N) = \sum_{k=0}^{N} k\mathbb{P}(X = k) \uparrow \sum_{k=0}^{\infty} k\mathbb{P}(X = k) \ ,$$

and since $E_{\mathbb{P}}(Y^{(N)}) \to E_{\mathbb{P}}(X)$, necessarily

$$E_{\mathbb{P}}(X) = \sum_{k=0}^{\infty} k\mathbb{P}(X = k) \ .$$

3. On an abstract probability space $(\Omega, \mathcal{F}, \mathbb{P})$,

   let $N(\omega)$ be a Poisson distributed random variable with parameter $\lambda > 0$, such that

$$\mathbb{P}(\{\omega : N(\omega) = k\}) = P_\lambda(\{k\}) = \exp(-\lambda)\frac{\lambda^k}{k!}$$

   (a) Check that $(P_\lambda(\{k\}) : k \in \mathbb{N})$ defines a probability distribution on $\mathbb{N} = \{0, 1, 2, \dots\}$, in particular that $P_\lambda(\mathbb{N}) = 1$.
   Solution :

$$P_\lambda(\mathbb{N}) = \sum_{k=0}^{\infty} \mathbb{P}_\lambda(\{k\}) = \exp(-\lambda)\sum_{k=0}^{\infty}\frac{\lambda^k}{k!} = \exp(-\lambda)\exp(\lambda) = 1$$

   (b) Compute the *moment generating function* $m : \mathbb{R} \to [0, \infty]$

$$m(\theta) = E_{\mathbb{P}}\big(\exp(\theta N)\big), \quad \theta \in \mathbb{R}.$$

   Solution :

$$E_{\mathbb{P}}\big(\exp(\theta N)\big) = \sum_{k=0}^{\infty}\exp(\theta k)\mathbb{P}_\lambda(\{k\}) = \exp(-\lambda)\sum_{k=0}^{\infty}\frac{\big(\lambda\exp(\theta)\big)^k}{k!} =$$
$$\exp(-\lambda)\exp\big(\lambda e^\theta\big) = \exp\big(\lambda(e^\theta - 1)\big)$$

   (c) Prove the following *Stein equation* for the Poisson distribution:

$$\lambda E_{\mathbb{P}}(g(N+1)) = E_{\mathbb{P}}(Ng(N))$$

   for every bounded sequence $(g_k : k \in \mathbb{N}) \subseteq \mathbb{R}$.
   Solution

$$\lambda E_{\mathbb{P}}(g(N+1)) = \lambda\exp(-\lambda)\sum_{n=0}^{\infty} g(k+1)\frac{\lambda^k}{k!}$$
$$= \lambda\exp(-\lambda)\sum_{k=1}^{\infty} g(k)\frac{\lambda^{k-1}}{(k-1)!} = \exp(-\lambda)\sum_{k=0}^{\infty} g(k)k\frac{\lambda^k}{k!} = E_{\mathbb{P}}(Ng(N))$$

   (d) Compute the expectations (moments) $\mathbb{E}_{\mathbb{P}}(N^q)$ for $q \in \mathbb{N}$.
   Solution We just give a recursive formula by using Stein equation. When $g_n = 1$

$$E_{\mathbb{P}}(N) = E_{\mathbb{P}}(Ng(N)) = E_{\mathbb{P}}(\lambda g(N+1)) = \lambda$$

When $g_n = n^q$, $q \geq 0$

$$E_{\mathbb{P}}(N^{q+1}) = E_{\mathbb{P}}(NN^q) = E_{\mathbb{P}}(Ng(N)) = \lambda E_{\mathbb{P}}(g(N+1))$$
$$= \lambda E_{\mathbb{P}}((N+1)^q) = \lambda \sum_{k=0}^{q} \binom{q}{k} E(N^k)$$

For example for $q = 1$, $E_{\mathbb{P}}(N^2) = \lambda(E_{\mathbb{P}}(N) + 1) = \lambda^2 + \lambda$.
For $q = 2$ $E_{\mathbb{P}}(N^3) = \lambda\big(E_{\mathbb{P}}(N^2) + 2E_{\mathbb{P}}(N) + 1\big) = \lambda^3 + 3\lambda^3 + \lambda$

(e) Compute the expectations $\mathbb{E}_{\mathbb{P}}\big(N^q \exp(\theta N)\big)$ for $\theta \in \mathbb{R}$ and $q \in \mathbb{N}$.
Solution For $q = 0$ we have already computed $E_{\mathbb{P}}(\exp(\theta N)) = \exp(\lambda(e^\theta - 1))$.
When $g_n = n^q \exp(\theta n)$, $q \geq 0$.

$$E_{\mathbb{P}}(N^{q+1} \exp(\theta N)) = E_{\mathbb{P}}(NN^q \exp(\theta N)) = E_{\mathbb{P}}(Ng(N))$$
$$= \lambda E_{\mathbb{P}}(g(N+1)) = \lambda E_{\mathbb{P}}((N+1)^q \exp(\theta N + 1))$$
$$= \lambda e^\theta E_\lambda((N+1)^q \exp(\theta N)) = \lambda e^\theta \sum_{k=0}^{q} \binom{q}{k} E_\lambda(N^k \exp(\theta N))$$

For example for $q = 1$ we get

$$E_{\mathbb{P}}(N \exp(\theta N)) = \lambda \exp(\lambda(e^\theta - 1) + \theta)$$

For $q = 2$

$$E_{\mathbb{P}}(N^2 \exp(\theta N)) = \lambda e^\theta E((N+1)\exp(\theta N)) = \lambda \exp(\lambda(e^\theta - 1) + \theta)(1 + \lambda e^\theta)$$

Alternatively, we could use a change of measure. Note that if $\psi > 0$ is the parameter of another Poisson distribution P_ψ, we have that $P_\psi \sim \Psi_\theta$ are equivalent (absolutely continuous with respect to each other) with likelihood ratio

$$Z(n) = \frac{dP_\psi}{dP_\lambda}(n) = \frac{P_\psi(\{n\}\ )}{P_\lambda(\{n\}\ )} = \left(\frac{\psi}{\lambda}\right)^n \exp(\lambda - \psi)$$

Therefore by the change of measure formula, for $\psi = \lambda e^{\theta}$

$$\mathbb{E}_{\lambda}\big(N^q \exp(\theta N)\big) = \mathbb{E}_{\lambda}\left(N^q \left(\frac{\psi}{\lambda}\right)^N\right) = \mathbb{E}_{\lambda}\left(N^q Z(N)\right) \exp\left(\lambda(e^{\theta} - 1)\right)$$
$$= \mathbb{E}_{\lambda}\left(N^q \frac{dP_{\psi}}{dP_{\lambda}}(N)\right) \exp\left(\lambda(e^{\theta} - 1)\right)$$
$$= \mathbb{E}_{\psi}\big(N^q\big) \exp\big(\lambda(e^{\theta} - 1)\Big) = \lambda \exp\big(\lambda(e^{\theta} - 1) + \theta\big) \sum_{k=0}^{q-1} \binom{q-1}{k} E_{\psi}\big(N^k\big)$$
$$= \lambda e^{\theta} \sum_{k=0}^{q-1} \binom{q-1}{k} E_{\lambda}\big(N^k \exp(\theta N)\big)$$