HU, Probability Theory Fall 2015, Solutions to Problems 4 (30.9.2015)

1. Prove that if $\mathbb{P}$ is a probability measure on $\Omega = \mathbb{R}^d$ equipped with the Borel σ-algebra $\mathcal{B}(\mathbb{R}^d)$, every Borel set $B \in \mathcal{B}(\mathbb{R}^d)$ satisfies the following **Approximation Property** : for every $\varepsilon > 0$ there is an open set $U \subseteq \mathbb{R}^d$ and a closed set $C \subseteq \mathbb{R}^d$ such that $U \supseteq B \supseteq C$ and $\mathbb{P}(U \setminus C) \leq \varepsilon$.

   To show that consider the class of events

   $$\mathcal{D} = \left\{ B \in \mathcal{B}(\mathbb{R}^d) \text{ which has the Approximation property } \right\} \subseteq \mathcal{B}(\mathbb{R}^d).$$

   - Show first that the class

     $$\mathcal{C} = \{ \ C \subseteq \mathbb{R}^d, \ C \text{ closed } \} \subset \mathcal{D}$$

     and it is a π-class (closed under intersections).

     Solution: $\mathcal{C}$ is a π-class since the arbitrary intersection of closed set is closed. Note that the finite union of closed sets is closed but the infinite union of closed sets if not always closed, for example

     $$\bigcup_{n \geq 1} [-1 + 1/n, 1 - 1/n] = (-1, 1) \text{ open}$$

     Hint if C is closed, let $C^\varepsilon = \{ \ y : \exists x \in C \text{ with } |x - y| < \varepsilon \} \supseteq C$ Show that C^ε is open and

     $$C = \bigcap_{n \in \mathbb{N}} C^{1/n}$$

     Use the σ-addivity of $\mathbb{P}$ to show that C has the Approximation propery.

     Solution C^ε is open: if $y \in C^\varepsilon$, there exists $x \in C$ with $|x - y| < \varepsilon$. Then $y \in B(x, \varepsilon) = \{z \in \mathbb{R}^d : |x - z| < \varepsilon\} \subseteq C^\varepsilon$, where $B(x, \varepsilon)$ is an open ball.

     Therefore we have $C^{1/n} \supseteq C^{1/(n+1)} \supseteq C$ with $C^{1/n}$ open and C closed.

     Moreover

     $$C = \bigcap_{n \in \mathbb{N}} C^{1/n}$$

since $C^{1/n} \supseteq C$ $\forall n$, and if $y \in \bigcup_{n\in\mathbb{N}} C^{1/n}$, there is a sequence $\{x_n :\in \mathbb{N}\} \subseteq C$ with $|x_n - y| < 1/n$. Therefore $\lim_{n\to\infty} x_n = y$, and since C is closed necessarily $y \in C$.

Consider now the sequence of events $(C_n \setminus C) \downarrow \emptyset$. Note that $C_n \setminus C = C_n \cap C^c$ is an open set, in particular it is Borel measurable.

Since $\mathbb{P}$ is σ-additive, $\mathbb{P}(C_n \setminus) \downarrow 0$, which means $P(C_n \setminus C) < \varepsilon$ for n large enoufg that all the closed set have the approximation property and are in the class $\mathcal{D}$.

- Then show that $\mathcal{D}$ is a Dynkin class.

  Solution We show directly that it is a σ-algebra. $\Omega = \mathbb{R}^d$ is both an open and a closed set, therefore we can take $U = \mathbb{R}^d = C$ to see that it has trivially the approximation property.

  When $A \in \mathcal{D}$, there are U open and C closed with $U \supseteq A \supseteq C$ and $\mathbb{P}(U \setminus C) < \varepsilon$. Then $U^c \subseteq A^c \subseteq C^c$ with C^c open and U^c closed, and $C^c \setminus U^c = C^c \cap (U^c)^c = C^c \cap U = U \setminus C$, so that $\mathbb{P}(C^c \setminus U^c) = \mathbb{P}(U \setminus C) = \varepsilon$.

  Let $(A_n : n \in \mathbb{N}) \subseteq \mathcal{D}$, with $A_n \cap A_m = \emptyset$ $\forall m \neq n$, and let $A = \bigcup_{n\in\mathbb{N}} A_n$.
  Since $\mathbb{P}$ is σ-additive, $\mathbb{P}(A) = \sum_{n\in\mathbb{N}} \mathbb{P}(A_n) \leq 1$ which implies that $\lim_{n\to\infty} \mathbb{P}(A_n) = 0$.
  Therefore $\forall\varepsilon > 0$ $\exists N_\varepsilon$ such that $\mathbb{P}(A_n) \leq \varepsilon$ $\forall n \geq N_\varepsilon$.
  Since $(A_n : n \in \mathbb{N}) \subseteq \mathcal{D}$, there are open sets U_n and closed sets C_n such that $U_n \supseteq A_n \supseteq C_n$ $\forall n$ and

$$\mathbb{P}(U_n \setminus C_n) \leq \varepsilon 2^{-n}$$

  Take now $U = \bigcup_{n\in\mathbb{N}} U_n$ which is open since the arbitrary union of open sets is open,
  Then

$$\begin{aligned} \mathbb{P}(U \setminus A) &= \mathbb{P}\left(\bigcup_{n\in\mathbb{N}} U_n \setminus \bigcup_{m\in\mathbb{N}} A_m\right) \leq \mathbb{P}\left(\bigcup_{n\in\mathbb{N}} U_n \setminus A_n\right) \\ &\leq \sum_{n\in\mathbb{N}} \mathbb{P}(U_n \setminus A_n) \leq \varepsilon \sum_{n\in\mathbb{N}} 2^{-n} = \varepsilon \end{aligned}$$

  Take also $C = \bigcup_{n=1}^{N} C_n$ which is closed since the finite union of closed sets is closed (infinite unions are not always closed), and

$U \supseteq A \supseteq C$. Then

$$\mathbb{P}(A \setminus C) = \mathbb{P}\Big(\bigcup_{n>N} A_n\Big) + \mathbb{P}\Big(\Big(\bigcup_{n\le N} A_n\Big) \setminus C_n\Big)$$

$$\le \sum_{n>N} \mathbb{P}(A_n) + \mathbb{P}\Big(\bigcup_{n\le N} A_n \setminus C_n\Big) \le \varepsilon + \varepsilon \sum_{n=1}^{N} 2^{-n} \le 2\varepsilon$$

Moreover $U \setminus C = (U \setminus A) \cup (A \setminus C)$ where the union is disjoint. $\mathbb{P}(U \setminus C) = \mathbb{P}(U \setminus A) + \mathbb{P}(A \setminus C) \le 3\varepsilon$.

- Use Dynkin lemma to conclude that all Borel sets have the Approximation property.

  Solutions $\mathcal{D} \subseteq \mathcal{B}(\mathbb{R}^d)$ by definition and $\mathcal{D}$ is a σ-algebra containing the closed sets, which means it must contain the σ-algebra generated by the closed sets, which is the Borel σ-algebra $\mathcal{B}(\mathbb{R}^d)$.

- Prove also that when $B \in \mathcal{B}(\mathbb{R}^d)$ is a Borel set, $\forall \varepsilon > 0$ one can find an open set U and a compact set K with $U \ge B \ge K$ and $\mathbb{P}(U \setminus K) < \varepsilon$.

  Consider the sequence of closed balls

$$K_n = \overline{B(0,n)} = \{\ x : |x| \le n\}$$

. These are compact, since in $\mathbb{R}^d$ a set is compact if and only if it is bounded and closed, and

$$\Omega = \mathbb{R}^d = \bigcup_{n\in\mathbb{N}} K_n$$

.

Since $K_n^c \downarrow \emptyset$, since $\mathbb{P}$ is σ-additive $\mathbb{P}(K_n^c) \downarrow 0$.
Let B a Borel set and given $\varepsilon > 0$ let be U an open and C a closed set such that $U \supseteq B \supseteq C$ and $\mathbb{P}(U \setminus C) < \varepsilon$.
Let n large enough such that $\mathbb{P}(K_n^c) < \varepsilon$, and let $K = C \cap K_n$. $K \subseteq B$ is compact because it is closed and bounded, and

$$\mathbb{P}(U \setminus K) = \mathbb{P}(U \cap K^c) = \mathbb{P}(U \cap (C^c \cup K_n^c)) = \mathbb{P}((U \cap C^c) \cup (U \cap K_n^c))$$
$$\le P(U \setminus C) + \mathbb{P}(K_n^c) < 2\varepsilon$$

Remark We have used the Approximation property of the Borel sets in the proof of Kolmogorov extension Theorem.

2. Let $f : \mathbb{R} \to \mathbb{R}$ a non-decreasing function, $f(s) \le f(t)$ when $s \le t$. Show that f is Borel measurable, which means that for every Borel set $B \in \mathcal{B}(\mathbb{R})$, the counterimage $f^{-1}(B) := \{ t : f(t) \in B\}$ is a Borel set.

   Solution Note that f is not invertible since it could be non-decreasing without being stricly increasing.

   Define the generalized inverse of f as

$$f^{inv}(t) = \sup\{x : f(x) \le t\}$$

   Then we have two possible situations: the counterimage of the interval $(-\infty, t]$ is either

   $f^{-1}((-\infty, t]) = \{x : f(x) \le t\} = (-\infty, f^{inv}(t)]$

   or

   $f^{-1}((-\infty, t]) = \{x : f(x) \le t\} = (-\infty, f^{inv}(t))$.

   In each case it is a Borel set.

   Now the class

$$\mathcal{D} := \{B \in \mathcal{B}(\mathbb{R}) : f^{-1}(B) \in \mathcal{B}(\mathbb{R})\} \supseteq \mathcal{C} = \{ \ (-\infty, t] : t \in \mathbb{R}\}$$

   contains $\mathcal{C}$ which is a π-system. It is easy to check that $\mathcal{D}$ is a Dynkin class, which by Dynkin lemma implies

$$\mathcal{B}(\mathbb{R}) = \sigma(\mathcal{C}) \subseteq \mathcal{D} \subseteq \mathcal{B}(\mathbb{R})$$

   .

3. On a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, let $(A_n : n \in \mathbb{N})$ be any sequence of pairwise disjoint events, which means $A_i \cap A_j = \emptyset$ when $i \ne j$. Show that $\lim_{n\to\infty} \mathbb{P}(A_n) = 0$.

   Solution by σ-additivity:

$$1 \ge \mathbb{P}\left(\bigcup_{n\in\mathbb{N}} A_n\right) = \sum_{n\in\mathbb{N}} \mathbb{P}(A_n)$$

   since the series has positive terms and it is convergent, necessarily $\lim_{n\to\infty} \mathbb{P}(An) = 0$.

4. On a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, let $(A_\alpha, \alpha \in I)$ be a family of pairwise disjoint events, indexed by an index set I. Show that if $\mathbb{P}(A_\alpha) > 0$ $\forall \alpha \in I$, then I must be countable.

Hint: show that $\forall n \in \mathbb{N}$ the set $\mathcal{I}_n := \{\alpha : \mathbb{P}(A_\alpha) > 1/n\}$ is finite .

Solution Suppose that $\mathcal{I}_n$ is infinite, containing a countable set $\mathcal{J}$. Then

$$1 \geq \mathbb{P}\Big(\bigcup_{j\in\mathcal{J}} A_j\Big) = \sum_{j\in\mathcal{J}} \mathbb{P}\big(A_j\big) \geq \frac{\#\mathcal{J}}{n} = +\infty$$

which gives a contradiction. Thereofore each $\mathcal{I}_n$ is finite, and

$$\mathcal{I} = \bigcup_{n\in\mathbb{N}} \mathcal{I}_n$$

is at most countable because it is the countable union of finite sets.

5. Let $\mathbb{P}$ a probability on $\Omega = \mathbb{R}$ equipped with the Borel σ-algebra $\mathcal{B}(\mathbb{R})$. We have shown the *cumulative distribution function* $F(t) = P\big((-\infty, t]\big)$ is right-continuous, which means

$$F(t+) = \lim_{u\downarrow t} F(u) = F(t) \quad \forall t \in \mathbb{R} \; .$$

Denote the jump size of F at t by $\Delta F(t) = F(t) - F(t-)$ where $F(t-) = \lim_{s\uparrow t} F(s)$ is the limit from the left.

(a) Show that $\mathbb{P}(\{t\}\ ) = \Delta F(t)$.
Solution The singleton has the representation $\{t\} = \bigcap_{n\in\mathbb{N}}(t - 1/n, t]$. In otherwords, $(t-1/n, t] \downarrow \{\ t\}$. and by σ-additivity

$$P(\{t\}) = \lim_{n\to\infty} \mathbb{P}((t-1/n, t]) = \lim_{n\to\infty} \big(F(t) - F(t-1/n)\big) = F(t) - F(t-)$$

where $F(t-) = \lim_{s\uparrow t} F(s)$ is the limit from the left.

(b) Show that the set of discontinuities $J = \Big\{\ t \in \mathbb{R} :\ \Delta F(t) > 0\Big\}$ is at most countable.
Solution Let

$$J_n = \{t \in \mathbb{R} : \Delta F(t) > 1/n\}$$

Since

$$F(+\infty) = 1 \geq \sum_{t\in J_n} \Delta F(t) \geq \frac{\#J_n}{n}$$

which implies that $\forall n\ J_n$ is a finite set. Therefore J is at most countable, because it is the countable union of finite sets.

6. Suppose a function $F : \mathbb{R} \to [0,1]$ is given by

$$F(t) = \sum_{n=1}^{\infty} 2^{-n} \mathbf{1}(t \geq 1/n)$$

(a) Show that $F(t)$ the cumulative distribution function of a probability P on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$.

Solution

$$F(-\infty) = F(0) = \sum_{n=1}^{\infty} 2^{-n} \mathbf{1}(0 \geq 1/n) = 0$$
$$F(+\infty) = F(1) = \sum_{n=1}^{\infty} 2^{-n} \mathbf{1}(1 \geq 1/n) = \sum_{n=1}^{\infty} 2^{-n} = \frac{1/2}{1-1/2} = 1$$

When $s \leq t$, $\mathbf{1}(s \geq 1/n) \leq \mathbf{1}(t \geq 1/n)$ and $F(s) \leq F(t)$.
Note that $F^{(N)}(u) = \sum_{n=1}^{N} 2^{-n} \mathbf{1}(u \geq 1/n)$ is non-decreasing with respect to N and non-increasing with repsect to u, therefore $F^{(N)}(u)$ is non-decreasing as $N \uparrow \infty$ and $u \downarrow t$ and we can switch the order of the limits to prove that $t \mapsto F(t)$ is right continuous:

$$\lim_{u \downarrow t} F(u) = \lim_{u \downarrow t} \lim_{N \uparrow \infty} F^{(N)}(u) = \lim_{N \uparrow \infty} \lim_{u \downarrow t} F^{(N)}(u) = \lim_{N \uparrow \infty} F^{(N)}(t) = F(t)$$

where the indicators of the $[1/n, \infty)$ intervals are right-continuous:

$$\lim_{u \downarrow t} \mathbf{1}_{[1/n,\infty)}(u) = \mathbf{1}_{[1/n,\infty)}(t) \ .$$

Note that P is discrete with probability mass $2^{-n} = F(n^{-1}) - F(n^{-1}-)$ at the points n^{-1} for each $n \in \mathbb{N}$.
For such P, compute the probabilities of the following events:

- $A = [1, \infty)$,
  Solution $P(A) = \sum_{n:1/n \in A} P(\{1/n\}) = \sum_{n:1/n \geq 1} P(\{1/n\}) = P(\{1\}) = 1/2$.
- $B = [1/10, \infty)$,
  Solution

  $$P(B) = \sum_{n:1/n \in B} P(\{1/n\}) = \sum_{n:1/n \geq 1/10} P(\{1/n\}) = \sum_{1 \leq n \leq 10} P(\{1/n\}) = \sum_{1 \leq n \leq 10} 2^{-n} = (1 - 2^{-10})$$

- $C = \{\ 0\}$, **Solution**

$$P(C) = \sum_{n:1/n \in C} P(\{1/n\}) = \sum_{n \in \emptyset} P(\{1/n\}) = 0$$

- $D = [0, 1/2)$,
  Solution

$$P(D) = \sum_{n:1/n \in D} P(\{1/n\}) = \sum_{n:1/n<1/2} P(\{1/n\}) = \sum_{n:n>2} P(\{1/n\})$$
$$= 1 - \sum_{n:n \le 2} P(\{1/n\}) = 1 - P(\{1\}) - P(\{1/2\}) = 1 - 1/2 - 1/4 = 1/4$$

- $E = (-\infty, 0)$
  Solution $P(E) = \sum_{n:1/n \in E} P(\{1/n\}) = \sum_{n \in \emptyset} P(\{1/n\}) = 0,$
- $G = (0, \infty)$.
  Solution $1 \le P(G) \ge P((0, 1]) = \sum_{n:1/n \in (0,1]} P(\{1/n\}) = \sum_{n \ge 1} 2^{-n} = 1$, and $P(G) = 1$.

(b) Define a random variable X on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ of your choice, with a probability $\mathbb{P}$ of your choice, such that the distribution $\mathbb{P}(\{\omega : X(\omega) \le t\}) = F(t)$.

Solution: you can always define the random variable as the identity map $X(\omega) = \omega$ in the space where it takes values, in this case $\mathbb{R}$ equipped with the Borel σ-algebra $\mathcal{B}(\mathbb{R})$.

For the distribution defined by the cumulative distribution function $F(t)$ above we can also take a random variable N on the space $\Omega = \mathbb{N} = \{1, 2, 3, \dots\}$ equipped with the probability measure $P(\{n\}) = 2^{-n}$, defined as the identity $N(n) = n$, and then take $X(n) = 1/N(n) = 1/n$.