HU, Probability Theory Fall 2015, Solutions to Problems 3 (23.9.2015)

(On the Cylinder algebra on an infinite product space).

Let S be an abstract probability space equipped with a σ-algebra $\mathcal{S}$, for example $S = \mathbb{R}^d$ and $\mathcal{S} = \mathcal{B}(\mathbb{R}^d)$, the Borel σ-algebra. and T an (infinite) arbitrary set . Consider the space $\Omega = S^T$, whose elements are the maps $\omega : T \to S$, with $t \mapsto \omega_t \in S$.

We can also understand Ω as the infinite product space $\Omega = \prod_{t \in T} S_t$, where each S_t is a copy of S.

A cylinder is an Ω-subset with representation

$$C = \{\omega : (\omega_{t_1}, \omega_{t_2} \dots, \omega_{t_d}) \in B_{t_1 \dots t_d}\} \tag{0.1}$$

for some $d \in \mathbb{N}$, $t_1, \dots, t_d \in T$ and $B_{t_1 \dots t_d} \in \mathcal{S}^{\otimes d} = \underbrace{\mathcal{S} \otimes \mathcal{S} \otimes \cdots \otimes \mathcal{S}}_{d\text{-times}}$, the d-fold product of σ-algebrae. In other words, whether a function ω belongs to a cylinder C or not it is determined by its values on a finite number of coordinates.

Note that the cylinder representation (0.1) is not unique, for example the same cylinder C could be expressed as

$$C = \{\omega : (\omega_{t_1}, \omega_{t_2} \dots, \omega_{t_d}, \omega_{t_{d+1}}) \in B_{t_1 \dots t_d} \times S\}$$

Q$_1$: Show that the cylinders $\mathcal{C} = \{C \subseteq \Omega : C \text{ is a cylinder }\}$ form an algebra of Ω-events.

Solution

Note that $\Omega = S^T \in \mathcal{C}$ since it has representation $S^T = \{\omega : (\omega_{t_1}, \dots, \omega_{t_d}) \in S^d\}$ for every $\{t_1, \dots, t_d\} \in T$ with $S^d \in \mathcal{S}^{\otimes d}$.

Consider C and $C\prime$ with representation

$$C\prime = \{\omega : (\omega_{s_1}, \omega_{s_2} \dots, \omega_{s_m}) \in A_{s_1 \dots s_m}\}$$

By taking as common set of time indexes the union of the time indexes $\{u_1, \dots, u_k\} := \{ t_1, \dots, t_d\} \cup \{ s_1, \dots, s_m\}$ and taking if necessary products of $B_{t_1,\dots,t_d}$ and $A_{s_1,\dots,s_m}$ with S^{k-d} and S^{k-m} one can represent C and $C\prime$ using the same time-indexes as

$$C = \{\omega : (\omega_{t_1}, \omega_{t_2} \dots, \omega_{u_d}, \omega_{u_k}) \in B_{u_1 \dots u_k} \times S\}$$

and

$$C\prime = \{\omega : (\omega_{t_1}, \omega_{t_2} \dots, \omega_{u_d}, \omega_{u_k}) \in A_{u_1 \dots u_k} \times S\}$$

with $B_{u_1,\dots,u_k}$ and $A_{u_1,\dots,u_k}$ in the product σ-algebra $\mathcal{S}^{\otimes k}$, and the union has representation

$$C \cup C\prime = \{\omega : (\omega_{t_1}, \omega_{t_2} \dots, \omega_{u_d}, \omega_{u_k}) \in A_{u_1 \dots u_k} \cup B_{u_1,\dots u_k} \times S\} \in \mathcal{C}$$

with $A_{u_1 \dots u_k} \cup B_{u_1,\dots u_k} \in \mathcal{S}^{\otimes k}$, and also the complement has representation

$$C^c = \Omega \setminus C = \{\omega : (\omega_{t_1}, \omega_{t_2} \dots, \omega_{u_d}, \omega_{u_k}) \in A^c_{u_1 \dots u_k}\} \in \mathcal{C}$$

since $A^c_{u_1 \dots u_k} = S^k \setminus A_{u_1 \dots u_k} \in \mathcal{S}^{\otimes k}$.

Q$_2$: However, the cylinders do not form a σ-algebra when T is infinite. Find an example where the countable intersection of cylinders is not a cylinder.

Solution Consider a sequence of distinct time-indexes $(t_n : n \in \mathbb{N}) \subset T$ and a sequence of sets $B_n \in \mathcal{S}$ with $B_n \neq S$ and $B_n \neq \emptyset$. For each n define the cylinder set

$$C^{(n)} = \{\omega \in S^T : \omega_{t_i} \in B_i \text{ for } 1 \leq i \leq n\}$$

Then it follows that

$$C^{(\infty)} := \bigcap_{n \in \mathbb{N}} C^{(n)} = \{\omega \in S^T : \omega_{t_i} \in B_i \ \forall i \in \mathbb{N}\}$$

is not a cylinder since it does not have a finite dimensional representation.

A consistent family $\mathcal{P}$ of finite dimensional distribution is a collection of probability measures $P_{t_1,\dots,t_d}$ on the respective product σ-algebrae $\mathcal{S}^{\otimes d}$ indexed by $t_1, t_2, \dots, t_d \in T$, where d varies in $\mathbb{N}$, satisfying the properties:

- $$P_{t_1,\dots,t_d}(B_{t_1} \times \dots \times B_{t_d}) = P_{t_{\pi(1)},\dots,t_{\pi(d)}}(B_{t_{\pi(1)}} \times \dots \times B_{t_{\pi(d)}}) =$$

  for every d, $t_1, \dots, t_d \in T$ and π permutation of $\{ 1, 2, \dots, d\}$, and $B_{t_i} \in \mathcal{S}$.

- $$P_{t_1,\dots,t_d}\big(B_{t_1,\dots,t_d}\big) = P_{t_1,\dots,t_d,t_{d+1}}\big(B_{t_1,\dots,t_d} \times S\big) =$$

  $\forall d,\ t_1,\dots,t_d,t_{d+1} \in T$ and $B_{t_1\dots,t_d} \in \mathcal{S}^{\otimes d}$.

Q$_3$: Show that the map

$$\mathbb{P}_0 : \mathcal{C} \to [0,1]$$

with $\mathbb{P}_0(C) = P_{t_1\dots t_d}(B_{t_1\dots t_d})$ for C with representation (0.1) is well defined, meaning that it does not depend on the particular representation of the cylinder C, and that $\mathbb{P}^0$ is finitely additive on the algebra $\mathcal{C}$.

Solution The consistency property of the family of finite dimensional distribution is just what we need to make sure that for a cylinder $C \in \mathcal{C}$ $\mathbb{P}_0(C)$ does not depend on the particular representation.

To show finite additivity, that $\mathbb{P}_0(C \cup C') = \mathbb{P}_0(C) + \mathbb{P}_0(C')$ when $C, C' \in \mathcal{C}$ with $C \cap C' = \emptyset$, just represent C and C' by using a common set of indexes $\{u_1,\dots,u_m\} \subset T$, and then use the addivity of the finite dimensional distribution $P_{u_1,\dots,u_m}$.

For each t, let Q_t a probability on $(S,\mathcal{S})$.

Define the family $\mathcal{Q}$ of finite dimensional distributions

$Q_{t_1\dots t_d} = Q_{t_1} \otimes Q_{t_2} \otimes \cdots \otimes Q_{t_d}$ as the product measure on the product space S^d equipped with product σ-algebra $\mathcal{S}^{\otimes d}$.

Q$_4$: Show that $\mathcal{Q}$ is a consistent family of finite dimensional distributions.

Solution For the product measure $Q_{t_1\dots t_d} = Q_{t_1} \otimes Q_{t_2} \otimes \cdots \otimes Q_{t_d}$ on $\mathcal{S}^{\otimes d}$ we have

- $$\begin{aligned} Q_{t_1,\dots,t_d}\big(B_{t_1} \times \cdots \times B_{t_d}\big) &= Q_{t_1}(B_{t_1}) \times Q_{t_d}(B_{t_d}) \times \cdots \times Q_{t_d}(B_{t_d}) \\ &= Q_{t_{\pi(1)},\dots,t_{\pi(d)}}\big(B_{t_{\pi(1)}} \times \cdots \times B_{t_{\pi(d)}}\big) = \end{aligned}$$

  for every $d,\ t_1,\dots,t_d \in T$ and π permutation of $\{\ 1,2,\dots,d\}$, and $B_{t_i} \in \mathcal{S}$.

- $$\begin{aligned} &Q_{t_1,\dots,t_d}\big(B_{t_1,\dots,t_d}\big) = Q_{t_1}(B_{t_1}) \times Q_{t_d}(B_{t_d}) \times \cdots \times Q_{t_d}(B_{t_d}) \times 1 \\ &= Q_{t_d}(B_{t_d}) \times \cdots \times Q_{t_d}(B_{t_d}) \times Q_{t_{d+1}}(S) = \\ &Q_{t_1,\dots,t_d,t_{d+1}}\big(B_{t_1,\dots,t_d} \times S\big) \end{aligned}$$

  $\forall d,\ t_1, \dots, t_d, t_{d+1} \in T$ and $B_{t_1 \dots, t_d} \in \mathcal{S}^{\otimes d}$.

Remark The next question which will be adressed in the lectures is: can we extend uniquely $\mathbb{P}^0$ to a σ-additive probability defined on the σ-algebra $\sigma(\mathcal{C})$ generated by the cylinders ? By Caratheordory theorem, it is enough to show that $\mathbb{P}^0$ is σ-additive on the cylinder algebra, namely if $(C_n : n \in \mathbb{N}) \subset \mathcal{C}$ is a cylinder sequence with $C_n \downarrow \emptyset$, necessarily $\mathbb{P}^0(C_n) \downarrow 0$. This is the content of Kolmogorov extension theorem, which requires an additional assumption on the probability space $(S, \mathcal{S})$.

$\mathbf{Q}_5$: In general, let Ω an abstract space and $\mathcal{E} \subseteq 2^\Omega$ a collection of Ω-subsets. Let $\mathcal{F} = \sigma(\mathcal{E})$ the σ-algebra generated by $\mathcal{E}$.

Show that $A \in \mathcal{F}$ if and only if $A \in \sigma(\mathcal{C})$ for some countable collection $\mathcal{C} \subseteq \mathcal{E}$, which may depend on A.

Solution : We show that the collection

$$\mathcal{D} := \big\{ A \in \mathcal{F} : A \in \sigma(\mathcal{C}) \text{ for some countable } \mathcal{C} \subseteq \mathcal{E} \ \big\}$$

is both a π-class and a Dynkin class and it contains $\mathcal{E}$.

First if A and B are in $\mathcal{D}$, then $A \in \sigma(\mathcal{C})$ and $B \in \sigma(\mathcal{C}')$ for some countable $\mathcal{C}, \mathcal{C}\prime \subseteq \mathcal{E}$. But then both $A, B \in \sigma(\mathcal{C} \cup \mathcal{C}')$ where $\mathcal{C} \cup \mathcal{C}' \subseteq \mathcal{E}$ is countable since it is the union of two countable sets, and also $A \cap B \in \sigma(\mathcal{C} \cup \mathcal{C}')$. Therefore $\mathcal{D}$ is a π-class.

$\mathcal{D}$ is also Dynkin class, because:

$\Omega \in \{\Omega, \emptyset\}$ which is the trivial σ-algebra,

when $A \supseteq B$ then $A \setminus B \in \sigma(\mathcal{C} \cup \mathcal{C}')$.

if $(A_n : n \in \mathbb{N}) \in \mathcal{D}$ and $A_n \uparrow A$, then for every $n \in \mathbb{N}$ $A_n \in \sigma(\mathcal{C}_n) \subseteq \mathcal{E}$, where C_n are countable sets, therefore $A = \bigcup_{n \in \mathbb{N}} A_n \in \sigma(C_n : n \in \mathbb{N})$ where $\bigcup_{n \in \mathbb{N}} C_n \subseteq \mathcal{E}$ is countable as countable union of countable sets and $A \in \mathcal{D}$.

On the other hand $\mathcal{D} \supseteq \mathcal{E}$, since for every $E \in \mathcal{E}$ $E \in \{\ \Omega, \emptyset, E, E^c\} \subseteq \mathcal{E}$, and $\{\ \Omega, \emptyset, E, E^c\}$ as a finite event algebra is trivially a countably generated σ-algebra.

Therefore $\mathcal{D}$ is a σ-algebra containing $\mathcal{E}$ and it must contain the σ-algebra $\mathcal{F} = \sigma(\mathcal{E})$ which is the smallest σ-algebra containing $\mathcal{E}$. Therefore $\mathcal{D} = \mathcal{F}$, and every set $A \in \mathcal{F}$ belongs to $\sigma(C_n : n \in \mathbb{N})$ for some event sequence $(C_n : n \in \mathbb{N}) \subseteq \mathcal{E}$,but the sequence may depend on the event A. When we can use the same sequence for all events $A \in \mathcal{F}$ we say that the σ-algebra $\mathcal{F}$ is countably generated. For example the Borel σ-algebra on $\mathbb{R}$ is countably generated since $\mathcal{B}(\mathbb{R}) = \sigma\big((-\infty, q] : q \in \mathbb{Q}\big)$.

Q$_6$: We come back to the construction of the σ-algebra generated by the cylinders on $\Omega = S^T$. Using the previous exercise, show that a set A in the σ-algebra $\sigma(\mathcal{C})$ generated by the cylinders is determined by at most countably many T-coordinates.

In particular, when $T = \mathbb{R}^m$ and $S = \mathbb{R}^d$, show that the space of continuous function

$$C(\mathbb{R}^m, \mathbb{R}^d) = \big\{\omega : \mathbb{R}^m \to \mathbb{R}^d \text{ continuous functions } \big\} \subseteq (\mathbb{R}^d)^{\mathbb{R}^m}$$

is not in the σ-algebra $\sigma(\mathcal{C})$ generated by the cylinders.

Solution By the previous exercise, if $A \in \mathcal{F} = \sigma(\mathcal{C})$ where $\mathcal{C}$ is the cylinder algebra, there is a sequence of cylinders $(C_n : n \in \mathbb{N}) \subseteq \mathcal{C}$ (which depends on A) such that $A \in \sigma(C_n : n \in \mathbb{N})$. Choose for each cylinder C_n in the sequence a finite dimensional representation on a finite time index set $I_n = \{t_1^n, \dots, t_{k_n}^n\} \subset T$. The countable union of those finite index sets $I = \bigcup_{n \in \mathbb{N}} I_n$ is countable, and whether $\omega = (\omega_t : t \in T)$ belongs to A or not is determined by the values $(\omega_t : t \in I)$ on the countable index set $I \subset T$.

We see for example that the set of continuous functions $C(\mathbb{R}^m, \mathbb{R}^d)$, is not in the σ-algebra $\sigma(\mathcal{C})$ since continuity has to be checked at all points $t \in T = \mathbb{R}^m$, and it is not enough to know the values $(\omega_t : t \in I)$ on a countable subsets of indexes $I \subset T$.