HU, Probability Theory Fall 2015, Problems 2 (16.9.2015)

1. A finitely additive probability $\mathbb{P}$ on a probability space Ω equipped with a σ-algebra $\mathcal{F}$ is also σ-additive if and only if for any event sequence $(A_n : n \in \mathbb{N}) \subseteq \mathcal{F}$ with $A_n \downarrow \emptyset$, meaning that $\forall n \in \mathbb{N}\ A_n \supseteq A_{n+1}$ and $\bigcap_{n\in\mathbb{N}} A_n = \emptyset$, it follows that $\mathbb{P}(A_n) \downarrow 0$.

   This does not hold for infinite measures with $\lambda(\Omega) = \infty$, for example for the Lebesgue measure λ on $\mathbb{R}$ equipped with the Borel σ-algebra $\mathcal{B}(\mathbb{R})$, such that $\lambda((a,b]) = (b-a)^+$. Here $x^+ = \max\{x, 0\}$ is the notation for the positive part of $x \in \mathbb{R}$.

   Find a counterexample, as a sequence $(A_n : n \in \mathbb{N}) \subset \mathcal{B}(\mathbb{R})$, with $A_n \downarrow 0$ but $\lambda(A_n) \not\to 0$.

   Solution Let λ be the Lebesgue measure on the measurable space $(\Omega, \mathcal{F})$ where $\Omega = \mathbb{R}$, $\mathcal{F} = \mathcal{B}(\mathbb{R})$, and $A_n = [n, \infty), \quad n \in \mathbb{N}$.

   Then $A_n \supseteq A_{n+1} \downarrow \bigcap_{n\in\mathbb{N}}[n, \infty) = \emptyset$

   but $\forall n\ \lambda(A_n) = +\infty$ which does not converge to zero.

2. Let $\Omega = \mathbb{R}^d$, the euclidean space. In general the Borel σ algebra is the smallest σ-algebra containing the open sets.

   For $t \in \mathbb{R}^d$, we introduce the infinite rectangle

   $$(-\infty, t] = \{s \in \mathbb{R}^d : \ s_i \le t_i, i = 1, \dots d\}$$

   Show that the class

   $$\mathcal{I} = \left\{(-\infty, q], q \in \mathbb{Q}^d\right\}$$

   is a π-class with $\sigma(\mathcal{I}) = \mathcal{B}(\mathbb{R}^d)$.

   Hint If U is open in $\mathbb{R}^d$, since $\mathbb{Q}$ is dense in $\mathbb{R}$, $\forall x \in U\ \exists r, q \in \mathbb{Q}^d$ such that $r < q$ (meaning that $r_i < q_i$ for each coordinate $i = 1, \dots, d$

   $$x \in (r, q) := (r_1, q_1) \times (r_2, q_2) \times \cdots \times (r_d, q_d) \subseteq U.$$

   i.e. there is a small open rectangle containg x which is contained in U.

   Solution $(-\infty t] \cap (-\infty, s] = (-\infty, t \wedge s]$ for $t, s \in \mathbb{R}$ where $(t \wedge s)_i = t_i \wedge s_i$, $i = 1, \dots, d$ and we denote $t_i \wedge s_i = \min\{t_i, s_i\}$. This implies that $\mathcal{I}$ is a π-system (closed under finite intersections).

Any rectangle

$$(s,t] = \prod_{i=1}^{d} (s_i, t_i] \subseteq \mathbb{R}^d$$

has representation

$(s,t] =$

$$(-\infty, t] \setminus \left(\bigcup_{i=1}^{d} \{(-\infty, t_1] \times \cdots \times (-\infty, t_{i-1}] \times (-\infty, s_i] \times (-\infty, t_{i+1}] \times \cdots \times (-\infty, t_i] \right),$$

which implies that $(s,t]$ belongs to the algebra $\mathcal{A}_0$ generated by the rectangles $\{(-\infty, t] : t \in \mathbb{R}^d\}$.

Note also that $(s,t) = \{x \in \mathbb{R}^d : s_i < x_i < t_i \ \forall 1 \le i \le d\} = \bigcup_{n \in \mathbb{N}} (s, t - \mathbf{1}/n]$ where $\mathbf{1} = (1, 1, \dots, 1) \in \mathbb{R}^d$, therefore (s,t) belongs to the generated σ-algebra $\mathcal{A} = \sigma(\mathcal{A}_0)$. and (s,t) is open in $\mathbb{R}^d$, which implies $\mathcal{A} \subseteq \mathcal{B}(\mathbb{R}^d) = \sigma(\{U \subset \mathbb{R}^d \text{ open set }\})$.

By using the hint, every open set is the countable union of rectangles $(r,q) \subseteq \mathcal{R}^d$ with $r \le q$, $r, q \in \mathbb{Q}^d$, which implies that $\sigma(\{\ (-\infty, q] : q \in \mathbb{Q}^d\}) \supseteq \mathcal{B}(\mathbb{R}^d)$ and the equality of these σ-algebrae follows.

3. Consider a probabilty triple $(\Omega, \mathcal{F}, \mathbb{P})$, and a sequence of events $A_n \in \mathcal{F}$ such that $\mathbb{P}(A_n) = 1 \ \forall n \in \mathbb{N}$.

   Show that $\mathbb{P}\left(\bigcap_{n \in \mathbb{N}} A_n\right) = 1$.

   Consider also a sequence $B_n \in \mathcal{F}$ such that $\mathbb{P}(B_n) = 0 \ \forall n \in \mathbb{N}$.

   Show that $\mathbb{P}\left(\bigcup_{n \in \mathbb{N}} B_n\right) = 0$.

   Solution For B_n with $\mathbb{P}(B_n) = 0$ Since $\mathbb{P}$ is additive for each finite $K \in \mathbb{N}$

$$0 \le \mathbb{P}\left(\bigcup_{n=1}^{K} B_n\right) \le \sum_{n=1}^{K} \mathbb{P}(B_n) = 0$$

   with equality when B_n are mutually disjoint but we do not assume that.

   Since $\mathbb{P}$ is σ-additive when we take the limit as $K \to \infty$ we can take the limit inside the probability

$$\mathbb{P}\left(\bigcup_{n=1}^{K} B_n\right) \le \sum_{n=1}^{K} \mathbb{P}(B_n)$$

$$0 \leq \mathbb{P}\left(\bigcup_{n=1}^{\infty} B_n\right) = \lim_{K\to\infty} \mathbb{P}\left(\bigcup_{n=1}^{K} B_n\right) \leq \lim_{K\to\infty} \sum_{n=1}^{K} \mathbb{P}(B_n) = \lim_{K\to\infty} \sum_{n=1}^{\infty} \mathbb{P}(B_n) = 0$$

If A_n is with $\mathbb{P}(A_n) = 1$, the complement has $\mathbb{P}(A_n^c) = 0$. Then by using the previous step and additivity of $\mathbb{P}$,

$$\bigcap_{n\in\mathbb{N}} A_n = \left(\bigcup_{n\in\mathbb{N}} A_n^c\right)^c$$

and by using additivity

$$\mathbb{P}\left(\bigcap_{n\in\mathbb{N}} A_n\right) = 1 - \mathbb{P}\left(\bigcup_{n\in\mathbb{N}} A_n^c\right) = 1 - 0$$

4. Consider the probability space $(\Omega, \mathcal{F})$ and an event sequence $(A_n : n \in \mathbb{N}) \subseteq \mathcal{F}$. We denote

$$\limsup_n A_n := \bigcap_{k\in\mathbb{N}} \bigcup_{n\geq k} A_k, \quad \liminf_n A_n := \bigcup_{k\in\mathbb{N}} \bigcap_{n\geq k} A_k,$$

   (a) Show that $(\limsup_n A_n) \in \mathcal{F}$.
   Solution Let $B_k = \bigcup_{n\geq k} A_k$. Since $\mathcal{F}$ is closed under countable unions $B_k \in \mathcal{F}$, and since $\mathcal{F}$ is closed under countable intersections we have also that $\limsup_n A_n = \left(\bigcap_{k\in\mathbb{N}} B_k\right) \in \mathcal{F}$. By taking complements and using the property from the next question we get that $\liminf_n A_n \in \mathcal{F}$ as well.

   (b) Show that $(\liminf_n A_n) = (\limsup_n A_n^c)^c$.
   where $B^c = \Omega \setminus B$ is the complement event.
   Solution

$$\left(\limsup_n A_n^c\right)^c = \left(\bigcap_{k\in\mathbb{N}} \bigcup_{n\geq k} A_k^c\right)^c = \bigcup_{k\in\mathbb{N}} \left(\bigcup_{n\geq k} A_k^c\right)^c =$$
$$\bigcup_{k\in\mathbb{N}} \bigcap_{n\geq k} (A_k^c)^c = \bigcup_{k\in\mathbb{N}} \bigcap_{n\geq k} A_k = \liminf_n A_n$$

   (c) Show that $(\liminf_n A_n) \in \mathcal{F}$.
   Solution By taking complements and using the properties from questions (1),(2) we get that $\liminf_n A_n \in \mathcal{F}$ as well.

(d) For $A \in \mathcal{A}$, let $\mathbf{1}_A(\omega)$, the indicator function of A defined for $\omega \in \Omega$. Show that

$$\mathbf{1}_{(\limsup_n A_n)}(\omega) = \limsup_n \mathbf{1}_{A_n}(\omega), \quad \mathbf{1}_{(\liminf_n A_n)}(\omega) = \liminf_n \mathbf{1}_{A_n}(\omega)$$

Solution Note that by definition for a sequence of functions $X_n : \Omega \to \mathbb{R}$ with $\omega \mapsto X_n(\omega)$

$$\limsup_n X_n(\omega) = \inf_{k\in\mathbb{N}} \sup_{n\geq k} X_n(\omega)$$

and

$$\liminf_n X_n(\omega) = \sup_{k\in\mathbb{N}} \inf_{n\geq k} X_n(\omega)$$

$\mathbf{1}_{(\limsup_n A_n)}(\omega) = 1 \iff \omega \in \limsup_n A_n$
$\iff \omega \in \bigcap_{k\in\mathbb{N}} \bigcup_{n\geq k} A_n$
$\iff \forall k \in \mathbb{N}\ \exists n = n(\omega, k)$ with $\omega \in A_n$
$\iff \forall k \in \mathbb{N}\ \exists n = n(\omega, k)$ with $\mathbf{1}_{A_n}(\omega) = 1$
and since the indicators can takes only the values 0 or 1 , this is equivalent to
$\iff \inf_{k\in\mathbb{N}} \sup_{n\geq k} \mathbf{1}_{A_n}(\omega) = 1$
$\iff \limsup_n \mathbf{1}_{A_n}(\omega) = 1$
By taking complements and using $\mathbf{1}_{A_n^c}(\omega) = 1 - \mathbf{1}_{A_n}(\omega)$ we obtain the corresponding equivalence for the liminf.

(e) Show that

$$\limsup_n A_n = \{\omega : \omega \in A_n \text{ infinitely often, which means for infinitely many } n \ \}$$

$$\liminf_n A_n = \{\omega : \omega \in A_n \text{ eventually, which means for all } n \text{ large enough } \}$$

Solution $\omega \in \limsup_n A_n \iff \omega \in \bigcap_k \bigcup_{n\geq k} A_n, \iff \forall k \in \mathbb{N} \exists n = n(\omega, k) \geq k$ with $\omega \in A_n \iff \omega \in A_n$ for infinitely many $n \in \mathbb{N}$ (which depend on the particular ω).
$\omega \in \liminf_n A_n \iff \omega \in \bigcup_k \bigcup_{n\geq k} A_n, \iff \exists k = k(\omega) \in \mathbb{N}$ such that $\forall n \geq k\ \omega \in A_n\ \forall n \geq k$, which means that $\omega \in A_n$ eventually (for all n which are large enough depending on ω).

(f) Let $A_n \subseteq B_n\ \forall n$. Show that

$$\limsup_n A_n \subseteq \limsup_n B_n, \quad \liminf_n A_n \subseteq \liminf B_n$$

Solution

$$\limsup_n A_n = \bigcap_k \bigcup_{n\geq k} A_n \subseteq \bigcap_k \bigcup_{n\geq k} B_n = \limsup_n B_n,$$
$$\liminf_n A_n = \bigcup_k \bigcap_{n\geq k} A_n \subseteq \bigcup_k \bigcap_{n\geq k} B_n = \liminf_n B_n.$$

(g) Suppose that for a probability $\mathbb{P}$ we have $\mathbb{P}(\limsup_n A_n) = 1$ and $\mathbb{P}(\liminf_n B_n) = 1$. Show that $\mathbb{P}(\limsup_n A_n \cap B_n) = 1$.

Solution $\mathbb{P}$ is additive, $P(E) = P(F) = 1$ implies $P(E \cap F) = 1$, therefore the event

$$D = \big\{\omega : \omega \in A_n \text{ infinitely often and } \omega \in B_n \text{ eventually } \big\}$$

has probability $\mathbb{P}(D) = 1$. On the other hand

$$D = \big\{\omega :\ \text{exists a sequence } (n_k \in \mathbb{N}) \text{ such that } \omega \in A_{n_k}\ \forall k\ \big\} \cap$$
$$\cap \big\{\ \omega :\ \text{exist } M \text{ such that } \omega \in B_j\ \forall j \geq M\ \big\}$$

by taking for each $\omega \in D$ the subsequence depending on ω given by
$(n'_k : k \in \mathbb{N}) = (n_k : k \in \mathbb{N}) \cap \{j : j \geq M\}$, we see that

$$D \subseteq D' := \big\{\omega :\ \text{exists a sequence } (n'_k \in \mathbb{N}) \text{ such that } \omega \in A_{n'_k} \cap B_{n_{k'}}\ \forall k\ \big\}$$

which implies $\mathbb{P}(D') = 1$. But $D' = \limsup_n A_n \cap B_n$.

5. Let $(\Omega, \mathcal{F})$ a probability space, with a sequence of probability measures $(\mathbb{P}_n : n \in \mathbb{N})$.

Suppose that $\forall A \in \mathcal{F}$ the limits

$$\mathbb{P}(A) := \lim_{n\to\infty} \mathbb{P}_n(A)$$

exists.

(a) Prove that in such case the map $\mathbb{P} : \mathcal{F} \longrightarrow [0,1]$ is a probability measure.

(b) For each event sequence $(A_k : k \in \mathbb{N}) \subseteq \mathcal{F}$ such that $A_k \downarrow \emptyset$ we have

$$\sup_{n\in\mathbb{N}} \mathbb{P}_n(A_k) \downarrow 0 \text{ as } k \uparrow \infty$$

Solution In fact this is a non-trivial result known as Hahn-Saks-Vitali Theorem.

One reference is from the book by J.L. Doob, Measure Theory Springer 1993, in paragraph III.10.

$\mathbb{P}(\Omega) = \lim_{n\to\infty} \mathbb{P}_n(\Omega) = 1$.

For $A, B \in \mathcal{F}$ with $A \cap B = \emptyset$

$$\mathbb{P}(A \cup B) = \lim_{n\to\infty} \mathbb{P}_n(A \cup B) = \lim_{n\to\infty} \big(\mathbb{P}_n(A) + \mathbb{P}_n(B)\big)$$
$$= \lim_{n\to\infty} \mathbb{P}_n(A) + \lim_{n\to\infty} \mathbb{P}_n(B) = \mathbb{P}(A) + \mathbb{P}(B)$$

Suppose that the event sequence $(A_n : n \in \mathbb{N}) \subseteq \mathcal{F}$ is such that $A_n \downarrow \emptyset$.

Then since $A_k \supseteq A_{k+1}$,

$$\mathbb{P}(A_k) = \lim_{n\to\infty} \mathbb{P}_n(A_k) \geq \lim_{n\to\infty} \mathbb{P}_n(A_{k+1}) = \mathbb{P}(A_{k+1})$$

and the sequence $(\mathbb{P}(A_k) : k \in \mathbb{N}) \subset [0,1]$ is non-decreasing and it has a monotone limits $q = \lim_{k\uparrow\infty} \mathbb{P}(A_k) \in [0,1]$.

To prove σ-additivity we have to show that $q = 0$.

By contradiction, assume $q > 0$, and define index-subsequences $(n_\ell : \ell \in \mathbb{N})$ and $(k_\ell : \ell \in \mathbb{N})$ in order to contradict the hypothesis.

Let $n_1 = k_1 = 1$. By induction assume we have already defined n_j and k_j for $1 \leq j \leq \ell$.

Let $A = \bigcap_k A_k$. Since $A_k \geq A_{k+1} \geq A$, and for all fixed k, $\lim_n P_n(A_k) \geq \lim_n P_n(A) = q > 0$,

we can choose first an $n_{\ell+1} > n_\ell$ large enough such that

$$P_{n_{\ell+1}}(A_{k_\ell}) \geq q7/8 > 0$$

and then, since for fixed n P_n is σ-additive and $A_k \downarrow \emptyset$, we can choose $k_{\ell+1} > k_\ell$ large enough such that

$$P_{n_{\ell+1}}(A_{k_{\ell+1}}) \leq q/8$$

Define

$$B_\ell = A_{k_\ell} \setminus A_{k_{\ell-1}}$$

then

$$P_{n_{\ell+1}}(B_\ell) \geq q3/4 > 0$$

For $\ell \geq 1$, j odd with $j > \ell$

$$P_{n_j}\Bigg(\bigcup_{\text{even } s\geq\ell} B_s\Bigg) \geq P_{n_j}(B_{j-1}) \geq q3/4 > 0$$

and by assumption the limit for fixed ℓ and $j \uparrow \infty$ exists and satisfies

$$P\Bigg(\bigcup_{\text{even } s\geq\ell} B_s\Bigg) \geq q3/4 > 0$$

Similarly

$$P\Bigg(\bigcup_{\text{odd } r\geq\ell} B_r\Bigg) \geq q3/4 > 0$$

Since $B_r \cap B_s = \emptyset$ for $r \neq s$,

$$A_{n_\ell} = \bigcup_{j\geq\ell} B_j = \Bigg(\bigcup_{\text{even } j\geq\ell} B_j\Bigg) \cup \Bigg(\bigcup_{\text{odd } j\geq\ell} B_j\Bigg)$$

where on the right hand side we have a disjoint union of two subsets, and P is finitely additive, it follows that $\forall \ell$

$$P(A_{n_\ell}) = P\Bigg(\bigcup_{r\geq i} B_r\Bigg) = P\Bigg(\bigcup_{\text{even } j\geq\ell} B_j\Bigg) + P\Bigg(\bigcup_{\text{odd } j\geq\ell} B_j\Bigg) \geq q3/2 > q > 0$$

with $n_\ell \to \infty$ which is contradiction with $P(A) = \lim P(A_n) = q$, unless $q = 0$ $\square$

For the (b) part, let $(A_k : k \in \mathbb{N}) \subseteq \mathcal{F}$ be an event sequence such that such that $A_k \downarrow \emptyset$. We have already proved that P is σ-additive

$$P(A_k) = \lim_{n\to\infty} \mathbb{P}_n(A_k) = \limsup_{n\in\mathbb{N}} \mathbb{P}_n(A_k) \downarrow 0 \text{ as } k \uparrow \infty$$

where by assumptions $P(A_k) = \lim_{n\to\infty} P_n(A_k)$ exists. Since for $a, b > 0$ $\max\{ a, b\} \leq (a + b)$,

$$\sup_n P_n(A_{k-1}) \geq \sup_n P_n(A_k) \leq \sum_{m\leq N} P_m(A_k) + \sup_{n\in N} P_n(A_k)$$

and for fixed N as $k \to \infty$

$$\lim_{k\to\infty} \sup_n P_n(A_k) \leq \lim_{k\to\infty} \sum_{m\leq N} P_m(A_k) + \lim_{k\to\infty} \sup_{n\geq N} P_n(A_k) = \lim_{k\to\infty} \sup_{n\in N} P_n(A_k), \quad \forall n \in \mathbb{N}$$

since $P_m(A_k) \to 0$ for $1 \leq k \leq N$ with N fixed. therefore

$$\lim_{k\to\infty} \sup_n P_n(A_k) \leq \lim_{N\to\infty} \lim_{k\to\infty} \sup_{n\geq N} P_n(A_k)$$

Now since $A_k \downarrow 0$, the doubly indexed sequence $a_k^N = \sup_{n\geq N} P_n(A_k)$ is non-increasing with respect to both k and N indexes, therefore it is possible to change the order of the limits (see Lemma 4.1.1.) in the lecture notes, and

$$\lim_{k\to\infty} \sup_n P_n(A_k) \leq \lim_{k\to\infty} \lim_{N\to\infty} \lim_{k\to\infty} \sup_{n\geq N} P_n(A_k)$$
$$= \lim_{k\to\infty} \limsup_n P_n(A_k) = \lim_{k\to\infty} \lim_n P_n(A_k) = \lim_{k\to\infty} P(A_k) = 0$$

where the limit is zero since we have already proven the σ-additivity.