HU, Probability Theory Fall 2015, Problems 1 (9.9.2015)

1. Let $\Omega = [0,1] \cap \mathbb{Q} = \{r \text{ rational} : 0 \leq r \leq 1 \}$,

   and $\mathcal{A}$ the collection of sets which can be represented as finite unions of intervals of type $(a,b] \cap \mathbb{Q}$, $[a,b] \cap \mathbb{Q}$,$(a,b) \cap \mathbb{Q}$, or $[a,b) \cap \mathbb{Q}$, with $0 \leq a \leq b \leq 1$.

   Define $\forall 0 \leq a \leq b \leq 1$

   $P_0((a,b] \cap \mathbb{Q}) = P_0([a,b] \cap \mathbb{Q}) = P_0((a,b) \cap \mathbb{Q}) = P_0([a,b) \cap \mathbb{Q}) = b - a,$

   - Show $\mathcal{A}$ is an algebra, which means $\Omega \in \mathcal{A}$, and when $A \in \mathcal{A}$ also $A^c := (\Omega \setminus A) \in \mathcal{A}$ and if $A, B \in \mathcal{A}$ also $A \cup B \in \mathcal{A}$.
   - Extend the function P_0 to a finitely additive probability on the algebra $\mathcal{A}$.
   - Show that such additive P_0 is not σ-additive.

   Hint $\Omega = [0,1] \cap \mathbb{Q}$ is countable !.

   Solution. If $A, B \subseteq [0,1]$ are finite union of intervals (which could be either open or closed on each side), also the complementa $A^c = [0,1] \setminus A$, $B^c = [0,1] \setminus B$ in $\Omega = [0,1]$ are finite union of intervals, and the intersection $A \cap B$ is a finite union of intervals. Since $A \cup B = (A^c \cap B^c)^c$, the same follows for finite union of intervals.

   The same properties hold after taking intersection withg the rationals $\mathbb{Q}$, which means that $\mathcal{A}$ is an algebra.

   Assume that $A \in \mathcal{A}$ has representation

   $$A = \bigcup_{i=1}^{n} \langle a_i, b_i \rangle \cap \mathbb{Q} \tag{0.1}$$

   where $0 \leq a_1 \leq b_1 \leq \cdots \leq a_n \leq b_n \leq 1$,

   and we use the same notation "$\langle$" for both open and closed parenthesis "(", "[",

   and the same notation "$\rangle$" for both "]", ")".

   Note that such representation is not unique, for example $[a,c] = [a,b) \cup [b,c] = [a,b] \cup [b,c] = [a,b] \cup (b,c]$.

   For A with is represented by disjoint rational intervals as in (0.1)

   $$P_0(A) = \sum_{i=1}^{n} (b_i - a_i)$$

P_0 is finitely additive but it cannot be countably additive since

$$\Omega = \bigcup_{q\in\mathbb{Q}\cap[0,1]} \{\, q\},$$

with $P_0(\{q\}) = q - q = 0$, while $P_0(\Omega) = P_0([0,1]) = 1 - 0 = 1$, which is contradiction with σ-additivity since

$$1 = P_0(\Omega) \neq \sum_{q\in[0,1]\cap\mathbb{Q}} P_0(\{q\}) \quad \square$$

2. Consider an abstract set Ω, and define the collection

$$\mathcal{A} = \Big\{ A \subseteq \Omega : \text{ either } A \text{ or its complement } A^c = \Omega \setminus A \text{ is finite } \Big\}$$

- Show that when $\#\Omega = \infty$ $\mathcal{A}$ is an algebra but it is not a σ-algebra.
- For $A \in \mathcal{A}$, define $Q(A) = 0$ when A is finite and $Q(A) = 1$ when A is infinite. Show that Q is finitely additive on $\mathcal{A}$ but not σ-additive.

Solution: Note that if $A_1, A_2 \in \mathcal{A}$ are infinite sets with finite complement, such that $Q(A_1) = Q(A_2) = 1$, necessarily $A \cap B \neq \emptyset$, because otherwise by taking complements we would get

$$\Omega = A^c \cup B^c$$

as union of finite sets is finite, which is contraddiction with the assumption $\#\Omega = \infty$.

Now if A_1 and A_2 are finite, $A_1 \cup A_2$ is finite and finite additivity is satisfied for

$$Q(A_1 \cup A_2) = 0 = 0 + 0 = Q(A_1) + Q(A_2)$$

If A_1 is finite and A_2^c is finite, then $A_1 \cup A_2$ is infinite with complement $A_1^c \cap A_2^c \subseteq A_1^c$ which is finite, and finite additivity is satisfied for

$$Q(A_1 \cup A_2) = 1 = 1 + 0 = Q(A_1) + Q(A_2)$$

If A_1^c and A_2^c are finite, then $A_1 \cap A_2 \neq \emptyset$. In this case

$$Q(A_1 \cup A_2) = 1 < (1 + 1) = Q(A_1) + Q(A_2),$$

but this does not contradict finite additivity since A_1 and A_2 are not disjoint.

3. Let Ω an abstract set and 2^Ω its power, which is the collection of subsets $A \subseteq \Omega$.

Define the symmetric difference of $A, B \subseteq \Omega$ as

$$A\Delta B = (A \cup B) \setminus (A \cap B) = \{\omega : \omega \in A \text{ or } \omega \in B \text{ but not in both } \}$$

Note that for indicators we have

$$\mathbf{1}_{A\cap B} = \mathbf{1}_A \mathbf{1}_B, \quad \mathbf{1}_{A\cup B} = \mathbf{1}_A + \mathbf{1}_B - \mathbf{1}_A\mathbf{1}_B$$
$$\mathbf{1}_{(A\Delta B)} = \big(\mathbf{1}_A + \mathbf{1}_B\big) \bmod 2 = \mathbf{1}_A + \mathbf{1}_B - 2\times \mathbf{1}_A\mathbf{1}_B = 1_A\mathbf{1}_{B^c} + \mathbf{1}_B\mathbf{1}_{A^c}$$

Show that 2^Ω is a **ring** with respect to the operations Δ (sum) and $\cap$ (product), which means

- Find an identity element with respect to the operation Δ.
  Solution The emptyset $\emptyset$ is the identity w.r.t. Δ since $A\Delta\emptyset = \emptyset\Delta A = A$, $\forall A \subseteq \Omega$.
- Find an identity element with respect to the operation $\cap$.
  Solution Ω is the identity w.r.t. $\cap$ since $A \cap \Omega = \Omega \cap A = A$, $\forall A \subseteq \Omega$.
- Show that every element $A \subset \Omega$ has an additive inverse.
  Solution $A\Delta A = \emptyset$, which means $A = (-A)$ is the inverse of itself with respect to Δ.
- Show that Δ is associative and the distributive property holds between Δ and $\cap$.
  We use the indicators: Note that

$$\mathbf{1}_{A\cap B} = \mathbf{1}_A\mathbf{1}_B, \quad \mathbf{1}_{A\cup B} = \mathbf{1}_A + \mathbf{1}_B - \mathbf{1}_A\mathbf{1}_B,$$
$$\text{and } \mathbf{1}_{A\Delta B} = \mathbf{1}_A + \mathbf{1}_B - 2\mathbf{1}_A\mathbf{1}_B$$

  The distributive property means

$$(A\Delta B)\cap C = \big((A\cup B)\cap C\big) \setminus (A\cap B\cap C)$$
$$= \big((A\cap C)\cup(B\cap C)\big) \setminus \big((A\cap C)\cap(B\cap C)\big) = (A\cap C)\Delta(B\cap C)$$

  For the associative property we use the indicators: Note that

$$\mathbf{1}_{A\cap B} = \mathbf{1}_A\mathbf{1}_B, \quad \mathbf{1}_{A\cup B} = \mathbf{1}_A + \mathbf{1}_B - \mathbf{1}_A\mathbf{1}_B,$$
$$\text{and } \mathbf{1}_{A\Delta B} = \mathbf{1}_A + \mathbf{1}_B - 2\mathbf{1}_A\mathbf{1}_B$$

we have

$$
\begin{aligned}
&\mathbf{1}_{(A\Delta B)\Delta C} = \mathbf{1}_{A\Delta B} + \mathbf{1}_C - 2\mathbf{1}_{A\Delta B}\mathbf{1}_C \\
&= \mathbf{1}_{A\Delta B}(1-2\mathbf{1}_C) + \mathbf{1}_C = \big(\mathbf{1}_A + \mathbf{1}_B - 2\mathbf{1}_A\mathbf{1}_B\big)(1-2\mathbf{1}_C) + \mathbf{1}_C \\
&= \mathbf{1}_A + \mathbf{1}_B + \mathbf{1}_C - 2\mathbf{1}_A\mathbf{1}_B - 2\mathbf{1}_A\mathbf{1}_C - 2\mathbf{1}_B\mathbf{1}_C + 4\mathbf{1}_A\mathbf{1}_B\mathbf{1}_C \\
&= \mathbf{1}_{(A\Delta C)\Delta B} = \mathbf{1}_{(B\Delta C)\Delta A}
\end{aligned}
$$

where expression in the second last line does not depend on the order we take the symmetric differences of A, B, C.

4. Consider an arbitrary collection of σ-algebrae $\{\mathcal{G}_\alpha : \alpha \in \mathcal{I}\}$ on the same set Ω.

   Show that the intersection of σ-algebrae

$$\mathcal{G} := \bigcap_{\alpha\in\mathcal{I}} \mathcal{G}_\alpha$$

   is a σ-algebra.

   Solution.

   - $\Omega, \emptyset \in \mathcal{G}_\alpha \forall\alpha \in \mathcal{I}$ since $\mathcal{G}_\alpha$ is a σ-algebra, therefore $\Omega, \emptyset \in \mathcal{G}$.
   - When $A \in \mathcal{G}_\alpha \forall\alpha \in \mathcal{I}$ also the complemnt $A^c \in \mathcal{G}_\alpha \forall\alpha \in \mathcal{I}$, therefore $A^c \in \mathcal{G}$ when $A \in \mathcal{G}$.
   - When $\{A_n : n \in \mathbb{N}\} \subseteq \mathcal{G}_\alpha \forall\alpha \in \mathcal{I}$, also $A = \bigcup_{n\in\mathbb{N}} A_n \in \mathcal{G}_\alpha \forall\alpha \in \mathcal{I}$, which means $A \in \mathcal{G}$ when $\{A_n : n \in \mathbb{N}\} \subseteq \mathcal{G}$.

5. About countable and uncountable sets:

   (a) Show that in the blackboard represented as $[0,1]^2$ there is place for an uncountable amount of mutually non-intersecting zero symbols 'O', (circles), where the circles can be also inside each other but they should not touch each other.

   (b) Show that on the blackboard $= [0,1]^2$ or on an infinite blackboard like $= \mathbb{R}^2$ there is place for at most a countable numbers of mutually non-intersecting '8' symbols, or ∞-symbols if you like, where the symbols can contain each other but the boundaries of different curves cannot touch each other.

   Solution

(a) Consider the circles $C_r = \{ (x, y) \in \mathbb{R}^2 : x^2 + y^2 = r^2\} \subset [0, 1]^2$ $0 < r < 1/2$. For $0 < r \neq t \leq 1/2$, $C_r \cap C_t = \emptyset$, with uncountable index set $[0, 1/2]$.

(b) Consider a collection of non-intersecting curves with the shape of the ∞ symbol, where possibly an 8-curve can lay in the region bounded by another 8-curve, without intersections.

For every 8-curve we can choose two points p and $q \in \mathbb{Q}^2$ such that each lies inside a different region delimited by the two differnt loops forming the curve, and choose the pair (p, q) as a label for this particular 8-curve.

No other 8-curve can have the same labels, it cannot contain both points p and q inside different loops without intersecting the 8-curve labels by p and q. Since each such 8-curve can be labeled by two points with rational coordinates, and no other 8-curve can have the same labels, and $\mathbb{Q}^2$ is countable, it follows that the set of 8-curves which we can draw on the plane without intersections must be countable.