UH Probability Theory I, autumn 2015, exam solutions(English) (4.11.2015)

You can choose whether to write the exam in English or in Finnish, the Finnish version of the same exam is also available!

Choose 4 problems out of the list $\{ 1, 2, 3, 4, 5\}$, and answer to all problems questions.

In the problems, all random variables are defined on a probability space $(\Omega, \mathcal{F}, P)$.

1. Let $X(\omega)$ be an $\mathbb{R}$-valued random variable on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. and let $F_X(t) = P\big(\{\omega : X(\omega) \le t\}\big)$ its cumulative distribution function.

   Prove the following properties

   (a) $F(+\infty) := \lim_{t\to+\infty} F(t) = 1$ and $F(-\infty) := \lim_{t\to-\infty} F(t) = 0$

   (b) F is non-decreasing, $F(s) \le F(t)$ when $s \le t$.

   (c) F is right continuous $F(t+) = \lim_{u\downarrow t} F(u) = F(t) \quad \forall t \in \mathbb{R}$.

   (d) the set of discontinuites

   $$\big\{t : \Delta F(t) = \big(F(t) - F(t-)\big) > 0\big\}$$

   where $F(t-) = \lim_{r\uparrow t} F(r)$ denotes the left limit, is at most countable.

   Hint: when you take limits, use the σ-additivity of the probability measure.

   Solutions:

   (a) $F(+\infty) = \lim_{n\uparrow\infty} F(n) = \lim_{n\uparrow\infty} P(X \le n) = P(X \in \mathbb{R})$, since $(-\infty, n] \uparrow \mathbb{R}$ and P is σ-additive.

   (b) $F(-\infty) = \lim_{n\uparrow\infty} F(-n) = \lim_{n\uparrow\infty} P(X \le -n) = P(X \in \emptyset) = 0$, since $(-\infty, -n] \downarrow \emptyset$ and P is σ-additive.

   (c) For $s \le t$ $(-\infty, s] \subseteq (-\infty t]$, and since P is additive, $F(s) = P(X \in (-\infty, s]) \le F(t) = P(X \in (-\infty, t])$

   (d) Let $F(t+) = \lim_{n\uparrow\infty} F(t+1/n) = \lim_{n\to\infty} P(X \in (-\infty, t+1/n]) = P(X \in (-\infty, t]) = F(t)$, since $(-\infty, t + 1/n] \downarrow \bigcap_{n\in\mathbb{N}}(-\infty, t + 1/n] = (-\infty, t]$ and P is σ-additive.

(e) To show that the set of discontinuities is at most countable, since $F(t)$ is non-decreasing with values in $[0,1]$,

$$1 = F(+\infty) - F(-\infty) \geq \sum_{t\in\mathbb{R}} \Delta F(t)$$

which implies

$$\#\big\{t : \Delta F(t) > 1/n\} \leq n\big\} \leq n < \infty$$

and

$$\big\{t : \Delta F(t) > 0\} \leq n\big\} = \bigcup_{n\in\mathbb{N}} \big\{t : \Delta F(t) > 1/n\} \leq n\big\}$$

is countable because it is the countable union of finite sets.

(f) We say that a sequence of random variables $(X_n : n \in \mathbb{N})$ converges stochastically (or also in probability) to a random variable X, and use the notation $X_n \stackrel{P}{\to} X$, when $\forall \eta > 0$

$$\lim_{n\to\infty} P\big(\{|X_n - X| > \eta\}\big) = 0$$

i. Prove that when $X_n \stackrel{L^1(P)}{\to} X$ in $L^1(P)$ norm,
meaning that $E_P\big(|X_n - X|\big) \to 0$, then $X_n \stackrel{P}{\to} X$ (in probability).
Solution By Chebychev inequality $\forall \eta > 0$,

$$|X_n(\omega) - X(\omega)| \geq \eta \mathbf{1}\big(|X_n(\omega) - X(\omega)| > \eta\big)$$

by taking expectation

$$P\big(|X_n(\omega) - X(\omega)| > \eta\big) \leq \frac{1}{\eta} E_P\big(|X_n(\omega) - X(\omega)|\big) \longrightarrow 0$$

ii. Let $X(\omega), \widetilde{X}(\omega)$ and $X_n(\omega), n \in \mathbb{N}$ random variables.

Prove the following statement: if $X_n(\omega) \to X(\omega)$ P-almost surely as $n \to \infty$,
and $X_n(\omega) \stackrel{L^1(P)}{\to} \widetilde{X}(\omega)$,
(which means $E_P(|X_n - X|) \to 0$ kun $n \to \infty$),
then it follows that $\widetilde{X}(\omega) = X(\omega)$ P-almost surely.

Solution Since convergence in $L^1(P)$-norm implies stochastic convergence, we can write $\forall K \in \mathbb{N}$,

$$P\big(|X - \widetilde{X}| \ > 1/K\big) = P\big(|X - X_n + X - \widetilde{X}| \ > 1/K\big)$$
$$\leq P\big(|X - X_n| + |X_n - \widetilde{X}| \ > 1/K\big)$$

$\forall n \in \mathbb{N}$, where the right hand side converges to zero by using the previous result and the result in the next problem. Since the lest hand side does not depend on n necessarily

$$P\big(|X - \widetilde{X}| > 1/K\big) = 0$$

and by taking the countable union of P-null sets we obtain

$$P\big(|X - \widetilde{X}| > O\big) = P\Big(\bigcup_{K\in\mathbb{N}} \{|X - \widetilde{X}| \ > 1/K\}\Big)$$
$$\leq \sum_{K\in\mathbb{N}} P\big(\{|X - \widetilde{X}| > 1/K\}\big) = 0$$

which means that $P\big(\{\ \omega : X(\omega) = \widetilde{X}(\omega)\big) = 1$

(g) Let $X, X_1, X_2, \dots X_n, \dots$ and $Y, Y_1, Y_2, \dots, X_n, \dots$ be random variables on the probability space $(\Omega, \mathcal{F}, P)$ such that
$X_n \xrightarrow{P} X$ and $Y_n \xrightarrow{P} Y$ (in probability).
Show that $(X_n + Y_n) \xrightarrow{P} (X + Y)$.
Hint for $a, b \in \mathbb{R}$, $|a + b| > \eta \Longrightarrow |a| > \eta/2$ or $|b| > \eta/2$
Solution For $\eta > 0$

$$P\big(|(X_n + Y_n) - (X + Y)| > \eta\big) = P\big(|(X_n - X) + (Y_n - Y)| > \eta\big)$$
$$\leq P\big(|X_n - X| + |Y_n - Y| > \eta\big)$$
$$\leq P\big(\{\ |X_n - X| > \eta/2\} \cup \{\ |Y_n - Y| > \eta/2\}\big)$$
$$\leq P\big(\ |X_n - X| > \eta/2\big) + P\big(\ |Y_n - Y| > \eta/2\big) \longrightarrow 0$$

as $n \to \infty$ since $X_n \xrightarrow{P} X$ and $Y_n \xrightarrow{P} Y$ (in probability).

(h) Let $U(\omega)$ and $V(\omega)$ be random variables such that $\forall s, t \in [0, 1]$,

$$P(U \leq s, V \leq t) = st$$

i. Show that U and V are independent and both uniformly distributed on $[0, 1]$.

ii. Let $X(\omega) = U(\omega)V(\omega)$. Show that the map $\omega \mapsto X(\omega)$ defines a random variable, and compute the cumulative distribution function of X

$$F_X(t) = P(X \le t) = P(UV \le t)$$

Hint: for any bounded Borel measurable function $g(u, v)$,

$$E_P\big(g(U, V)\big) = \int_0^1 \int_0^1 g(u, v) du dv$$

iii. Compute the probability density function of X.

iv. Compute the expectation $E_P(X)$ (Hint: you can compute first $E_P(U) = E_P(V)$ and use independence).

Solution The cumulative distribution function is given by

$$F_X(t) = P(X \le t)P(UV \le t) = \int_0^1 \int_0^1 \mathbf{1}(uv \le t) du dv = \int_0^1 \left(\int_0^{1\wedge(t/v)} du \right) dv =$$
$$\int_0^t \left(1 \wedge \frac{t}{v} \right) dv = \int_0^t 1 dv + \int_t^1 \frac{t}{v} dv = t - t \log t = (1 - \log t)t$$

for $t \in [0, 1]$, with $P(UV \le t) = 0$ for $t \le 0$ and $P(UV \le t) = 1$ for $t \ge 1$. Note that by l'Hospital rule

$$\begin{aligned} &\lim_{t\downarrow 0} F_X(t) = \lim_{t\downarrow 0}(1 - \log t)t \\ &= \lim_{t\downarrow 0} t - \lim_{t\downarrow 0} \frac{\log t}{1/t} = 0 + \lim_{t\downarrow 0} \frac{1/t}{1/t^2} = \lim_{t\downarrow 0} t = 0 = F_X(0) \end{aligned}$$

The density function is the derivative of the cumulative distribution function

$$p_X(t) = \frac{d}{dt} F_X(t) = \frac{d}{dt}\big((1 - \log t)t\big) = -\log t \qquad \text{for } t \in [0, 1], \, 0 \text{ otherwise}$$

Note that $p_X(0) = \lim_{t\downarrow 0} p_X(t) = +\infty$ and this probability density is unbounded. We can check that it is really a probability density: $-\log t \ge 0$ for $t \in [0, 1]$ and

$$\int_0^1 (-\log t) dt = \int_0^1 t d \log t = \int_0^1 t \frac{d \log t}{dt} dt = \int_0^1 t t^{-1} dt = \int_0^1 1 dt = 1$$

where we used the integration by parts formula. Since the $X = UV$ with U and V P-independent, and the expectation of the product of independent random variables is the product of the expectations,

$$E_P(X) = E_P(UV) = E_P(U)E_P(V) = E_P(U)^2 = 1/4$$

where

$$E_P(U) = \int_0^1 u du = \frac{1}{2}$$

We may also compute $E_P(X)$ directly using the density $p_X(t)$:

$$E_P(X) = \int_0^1 t p_X(t) dt = \int_0^1 t(-\log t) dt = -\frac{1}{2}\int_0^1 \log t dt^2 =$$
$$\frac{1}{2}\int_0^t t^2 d\log(t) = \frac{1}{2}\int_0^t t^2 t^{-1} dt = \frac{1}{2}\int_0^1 t dt = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$$

where we used the integration by parts formula.

(i) Consider a sequence $(X_n(\omega) : n \in \mathbb{N})$ of P-independent random variables which are identically distributed, with $P(X_n(\omega) \geq 0) = 1$ and $E_P(X_n) = \infty$.
This implies that $\forall K > 0$

$$\sum_{n=1}^{\infty} P(|X_n| > Kn) = \sum_{n=1}^{\infty} P(|X_1| > Kn) = \infty$$

(it follows by Fubini Theorem but you do not need to prove it now, since it was not in this part of the program)
Use Borel-Cantelli lemma (which one ?) to show that P-almost surely

$$\limsup_{n\to\infty} \frac{X_1(\omega) + X_2(\omega) + \cdots + X_{n-1}(\omega) + X_n(\omega)}{n} = +\infty$$

Solution Since the random variables X_n are independent, for each fixed $K > 0$ the events $A_n^K = \{\omega : X_n(\omega)/n > K\}$ are independent. By assumption $\sum_{n=1}^{\infty} P(A_n^K) = +\infty$ and the second Borel Cantelli lemma under independence of the event sequence applies, so that

$$P\left(\limsup_n A_n^K\right) = 1 \quad \forall K > 0$$

in other words

$$P\Big(\{\ \omega : X_n(\omega)/n > K \text{ for infinitely many indexes } n\ \}\Big) = 1$$

But this means that for each $K \in \mathbb{N}$, with probability 1

$$\limsup_n \frac{X_n(\omega)}{n} > K,$$

and since the countable intersection of P-almost sure events is a P-almost sure event, this implies that with probability 1

$$\limsup_{n\to\infty} \frac{X_1(\omega) + X_2(\omega) + \cdots + X_{n-1}(\omega) + X_n(\omega)}{n} \geq \limsup_n \frac{X_n(\omega)}{n} = +\infty$$

where all random variables $X_n(\omega)$ are non-negative with probability 1.