UH Probability Theory I, autumn 2015, exam (English) (29.11.2015)

You can choose whether to write the exam in English or in Finnish, the Finnish version of the same exam is also available!

Choose 4 problems out of the list $\{ 1,2,3,4,5\}$, and answer to all problems questions.

In the problems, all random variables are defined on a probability space $(\Omega, \mathcal{F}, P)$.

1. Recall the monotone convergence Theorem :

   If a sequence of non-negative random variables converges monotonically to a limit, $0 \leq X_n(\omega) \uparrow X(\omega)$ P-almost surely as $n \to \infty$, it follows that the expectation are converging monotonically to the expectation of the limit: $E_P(X_n) \uparrow E_P(X) \in [0, +\infty]$.

   (a) Prove Fatou lemma by using the monotone convergence Theorem: for a sequence of non-negative random variables $X_n(\omega) \geq 0$ P-melkein varmasti $\forall n \in \mathbb{N}$, we have

   $$E_P\big(\liminf_n X_n\big) \leq \liminf_n E_P\big(X_n\big)$$

   Solution Fatou Lemma is proved in your favourite probability theory textbook and in the course lecture notes (Lemma 4.1.4).

   (b) The reverse Fatou lemma is about the$\limsup_n X_n(\omega)$, stating that (under some assumptions on the sequence)

   $$E_P\big(\limsup_n X_n\big) \geq \limsup_n E_P(X_n)$$

   Under which assumptions on X_n this is true ? Prove the reverse Fatou lemma. **Solution** This does not hold without additional assumptions (lemma 4.1.5 in the lecture notes) A sufficient condition is that thre is an upper bound $Y(\omega) \in L^1(P)$ such that

   $$|X_n(\omega)| \leq Y(\omega) \qquad P\text{-almost surely } \forall n \in \mathbb{N}$$

   (c) Let $\{ X_n(\omega) : n \in \mathbb{N}\}$ and $X(\omega)$ random variables such that

   $$\lim_{n \to\infty} X_n(\omega) = X(\omega) \qquad P\text{-almost surely} \tag{0.1}$$

Give a sufficient condition for the convergence of the expectations $E_P(X_n) \to E_P(X)$. **Solution.** By using Lebesgue dominated convergence Theorem , which states that $E(X_n) \to E(X)$ when $X_n(\omega) \to X(\omega)$ P-almost surely and there is an integrable upper bound $Y \in L^1(P)$ such that

$$|X_n(\omega)| \le Y(\omega) \qquad P\text{-almost surely } \forall n \in \mathbb{N}$$

Show also a counterexample where $\lim_{n \to \infty} X_n(\omega) = X(\omega)$ P-almost surely but $E_P(X_n)$ does not converge towards $E_P(X)$.
Solution. Check example(4.1.1) in the lecture notes.

2. On a probability space $(\Omega, \mathcal{F}, P)$, let $G(\omega)$ be a standard Gaussian random variable with cumulative disrtibution function

$$\Phi(t) = P(G \le t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} \exp\Big(-\frac{x^2}{2}\Big) dx \qquad (0.2)$$

(a) Compute the expectation $E_P\Big(\exp\big(G^2\lambda/2\big)\Big) \in [0, +\infty]$ for $\lambda \in \mathbb{R}$.

Hint Since $\Phi(t)$ is the cumulative distribution function of a probability, $\Phi(+\infty) = 1$ and

$$\int_{-\infty}^{+\infty} \exp\Big(-\frac{x^2}{2}\Big) dx = \sqrt{2\pi} \ . \qquad (0.3)$$

Solution.

$$E_P\Big(\exp\big(G^2\lambda/2\big)\Big) = \int_{\mathbb{R}} \exp\big(x^2\lambda/2\big) P(G \in dx) =$$

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} \exp\big(x^2\lambda/2\big) \exp\Big(-\frac{x^2}{2}\Big) dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} \exp\Big(-\frac{x^2(1-\lambda)}{2}\Big) dx$$

which is $+\infty$ when $\lambda \ge 1$, and for $\lambda < 1$, by the chamnge of variable $y = x\sqrt{1-\lambda}$,

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} \exp\Big(-\frac{y^2}{2}\Big) \Big|\frac{dx}{dy}\Big| dy = (1-\lambda)^{-1/2} P(G \in \mathbb{R}) = (1-\lambda)^{-1/2}$$

(b) Compute the upper bound on the right side of the Chentsov inequality below.

$$P(|G| \ge t) = P\bigg(\exp\Big(\frac{\lambda\, G^2}{2}\Big) \ge \exp\Big(\frac{\lambda\, t^2}{2}\Big)\bigg) \qquad \forall \lambda > 0$$

$$\Longrightarrow P(|G| \ge t) \le \inf_{\lambda \in \mathbb{R}} \bigg\{ \exp\Big(-\frac{\lambda t^2}{2}\Big) E_P\Big(\exp\Big(\frac{\lambda G^2}{2}\Big)\Big)\bigg\}$$

Hint Find the infimum of the function by differentiating the function or its logarithm.

Solution. We minimize w.r.t. λ the logarithm

$$\lambda \mapsto -\frac{\lambda t^2}{2} - \frac{1}{2}\log(1-\lambda)$$

Since when the minimum is achieved at λ_* the derivative should be zero whenever it exists, we find that $\lambda_* = 1 - t^{-2}$, and corresponding best upper bound is given by

$$\inf_{\lambda \in \mathbb{R}} \left\{ \exp\left(-\frac{\lambda t^2}{2}\right)(1-\lambda)^{-1/2} \right\} = \exp\left(-\frac{\lambda_* t^2}{2}\right)(1-\lambda_*)^{-1/2} = \exp\left(\frac{1-t^2}{2}\right)t$$

$$P(|G| \geq t) \leq \min\left\{1, \exp\left(\frac{1-t^2}{2}\right)t\right\}$$

3. Let $\varepsilon > 0$, and $(X_n(\omega) \in \mathbb{N})$ a sequence of random variables (not necessarily P-independent !) such that

$$P\left(X_n = (n^{(1+\varepsilon)} - 1)\right) = n^{-(1+\varepsilon)} = 1 - P\big(X_n = -1\big)$$

We can interpret X_n as the reward of a gambler in a lottery, where the lottery ticket costs 1 €, giving the possibility to win $n^{(1+\varepsilon)}$ € with probability $n^{-(1+\varepsilon)}$.

(a) Check that $E_P(X_n) = 0$. This means in that the game is "fair".

Solution

$$E_P(X_n) = (n^{(1+\varepsilon)} - 1)P(X_n = (n^{(1+\varepsilon)} - 1) - P(X_n = -1) =$$
$$\big(n^{(1+\varepsilon)} - 1\big)n^{-(1+\varepsilon)} - \big(1 - n^{-(1+\varepsilon)}\big) = 0$$

(b) Prove :

$$\sum_{n=1}^{\infty} n^{-(1+\varepsilon)} \begin{cases} = \infty & \text{for } \varepsilon \leq 0 \\ < \infty & \text{for } \varepsilon > 0 \end{cases}$$

Solution

$$\sum_{n=2}^{\infty} n^{-(1+\varepsilon)} \leq \int_1^{\infty} x^{-(1+\varepsilon)} dx \leq \sum_{n=1}^{\infty} x^{-(1+\varepsilon)} \quad \forall \varepsilon \in \mathbb{R}$$

which implies that

$$\sum_{n=1}^{\infty} n^{-(1+\varepsilon)} < \infty \Longleftrightarrow \int_1^{\infty} x^{-(1+\varepsilon)} dx < \infty$$

where the integral on the right side is given by:

$$\int_1^t x^{-(1+\varepsilon)} dx = \begin{cases} \varepsilon^{-1}(1 - t^{-\varepsilon}) & \text{for } \varepsilon \neq 0 \\ \log(t) - \log(1) = \log(t) & \text{for } \varepsilon = 0 \end{cases}$$

Therefore

$$\int_1^{\infty} x^{-(1+\varepsilon)} dx = \lim_{t\to\infty} \int_1^t x^{-(1+\varepsilon)} dx = \begin{cases} \varepsilon^{-1} < \infty & \text{kun } \varepsilon > 0 \\ +\infty & \text{for } \varepsilon \leq 0 \end{cases}$$

(c) Let $S_n(\omega) = X_1(\omega) + X_2(\omega) + \cdots + X_n(\omega)$, the total profit of the gambler after n games, (negative values correspond to a loss). Show that with probabiliy $P = 1$

$$\lim_{n\to\infty} \frac{S_n(\omega)}{n} = -1$$

Hint: use Borel Cantellin lemma (which one, first or second ?). Note that while the game is fair, a gambler which goes on playing will eventually lose an infinite amount of money...!

Solution By assumption $\varepsilon > 0$, and

$$\sum_{n\in\mathbb{N}} P(X_n \neq -1) = \sum_{n\in\mathbb{N}} n^{-(1+\varepsilon)} < \infty$$

which by the first Borel Cantelli lemma implies

$$P\big(\limsup_n \{\ \omega :\ X_n(\omega) \neq -1\}\big) = 0$$

and

$$P\big(\liminf_n \{\ \omega :\ X_n(\omega) = -1\}\big) = 1$$

It means that, P-almost surely , there is $N(\omega) < \infty$ such that $X_n(\omega) = -1\ \forall n > N(\omega)$.

This implies P-almost surely, $\forall n > N(\omega)$

$$\frac{S_n(\omega)}{n} = \frac{S_{N(\omega)}(\omega)}{n} - \frac{n - N(\omega)}{n}$$

which implies

$$\lim_{n\to\infty} \frac{S_n(\omega)}{n} = \lim_{n\to\infty} \frac{S_{N(\omega)}(\omega)}{n} - \lim_{n\to\infty} \frac{n - N(\omega)}{n} = 0 - 1$$

4. We recall the definition of stochastic convergence (also called convergence in probability) for a sequence of random variables $(X_n(\omega) : n \in \mathbb{N})$:

$$X_n \xrightarrow{P} 0 \quad \Longleftrightarrow \forall \eta > 0, \lim_{n\to\infty} P\big(|X_n| > \eta\big) = 0$$

Show:

(a) If $\lim_{n\to\infty} X_n(\omega) = 0$ P-almost surely, it follows that $X_n \xrightarrow{P} 0$ (in probability). **Solution.** See Proposition (6.1.1.1) in the lecture notes

(b) If $X_n \xrightarrow{P} 0$ (in probability), there exists a deterministic subsquence $(n_k : k \in \mathbb{N})$ such that $\lim_{k\to\infty} X_{n_k}(\omega) = 0$ P-almost surely.

Hint Remember the Borel Cantelli lemma.

Solution. See Proposition (6.1.1.2) in the lecture notes.

(c) If $X_n \xrightarrow{L^q} 0$, meaning that $\lim_{n\to\infty} E_P\big(|X_n|^q\big) = 0$ with $q > 0$, it follows $X_n \xrightarrow{P} 0$ (in probability).

Hint Remember Chebychev inequality.

Solution . See page 85-85 in the Lecture notes.

(d)

$$X_n \xrightarrow{P} 0 \text{ (in probability)} \Longleftrightarrow d(X_n, 0) := E_P\left(\frac{|X_n|}{1+|X_n|}\right) \longrightarrow 0 \text{ as } n \to \infty$$

Hint The map $f : \mathbb{R}^+ \to [0,1]$ with $f(x) = x/(1+x)$ is strictly increasing.

Solution. See Theorem (6.1.1) in the lecture notes.

5. On an abstract probability space $(\Omega, \mathcal{F}, \mathbb{P})$,

let $N(\omega)$ be a Poisson distributed random variable with parameter $\lambda > 0$, such that

$$\mathbb{P}(\{\omega : N(\omega) = k\}) = P_\lambda(\{k\}) = \exp(-\lambda)\frac{\lambda^k}{k!}$$

(a) Check that $(P_\lambda(\{k\}) : k \in \mathbb{N})$ defines a probability distribution on $\mathbb{N} = \{0, 1, 2, \dots\}$, in particular that $P_\lambda(\mathbb{N}) = 1$.

Solution :

$$P_\lambda(\mathbb{N}) = \sum_{k=0}^{\infty} \mathbb{P}_\lambda(\{k\}) = \exp(-\lambda)\sum_{k=0}^{\infty}\frac{\lambda^k}{k!} = \exp(-\lambda)\exp(\lambda) = 1$$

(b) Compute the *moment generating function* $m : \mathbb{R} \to [0, \infty]$

$$m(\theta) = E_{\mathbb{P}}\big(\exp(\theta N)\big), \quad \theta \in \mathbb{R}.$$

Solution :

$$E_{\mathbb{P}}\big(\exp(\theta N)\big) = \sum_{k=0}^{\infty} \exp(\theta k)\mathbb{P}_\lambda(\{k\}) = \exp(-\lambda) \sum_{k=0}^{\infty} \frac{\big(\lambda \exp(\theta)\big)^k}{k!} =$$
$$\exp(-\lambda)\exp\big(\lambda e^\theta\big) = \exp\big(\lambda(e^\theta - 1)\big)$$

(c) Prove the following *Stein equation* for the Poisson distribution:

$$\lambda E_{\mathbb{P}}(g(N+1)) = E_{\mathbb{P}}(Ng(N))$$

for a sequence $(g(k) : k \in \mathbb{N}) \subseteq \mathbb{R}$.
Solution

$$\lambda E_{\mathbb{P}}(g(N+1)) = \lambda \exp(-\lambda) \sum_{n=0}^{\infty} g(k+1) \frac{\lambda^k}{k!}$$
$$= \lambda \exp(-\lambda) \sum_{k=1}^{\infty} g(k) \frac{\lambda^{k-1}}{(k-1)!} = \exp(-\lambda) \sum_{k=0}^{\infty} g(k) k \frac{\lambda^k}{k!} = E_{\mathbb{P}}(Ng(N))$$