HU, Probability Theory Fall 2015, Problems 7 (28.10.2015)
In the problems all random variables live in a probability space $(\Omega, \mathcal{F}, P)$.

1. On a probability space $(\Omega, \mathcal{F}, P)$,

   let $(X_n(\omega) : n \in \mathbb{N})$ be a sequence of exponential random variables such that

   $$P\big(X_1 > t_1, \dots, X_n > t_n\big) = \exp\left(-\lambda \sum_{i=1}^{n} t_n\right) \qquad \forall n \in \mathbb{N}, t_1, \dots, t_n \geq 0,$$

   where $\lambda > 0$ is a parameter.

   (a) Show that the random variables $(X_n(\omega) : n \in \mathbb{N})$ are independent under P.

   (b) Let

   $$Y_n(\omega) := \min\big\{X_1(\omega), X_2(\omega), \dots, X_n(\omega)\big\} \ .$$

   Compute $P(Y_n > t)$, and compute also the probability density function of Y_n .

   (c) Let $X_n^*(\omega) = \max\big\{X_1(\omega), X_2(\omega), \dots, X_n(\omega)\big\}$
   Compute $P(X_n^* \leq t)$. Compute also the probability density function of X_n^*.

   (d) Compute $\lim\limits_{n\to\infty} P\left(\lambda X_n^* \leq t + \log(n)\right)$.

   Hint: $\big(1 + x/n\big)^n \longrightarrow \exp(x)$ as $n \to \infty$.

2. Consider a sequence of random variables $(U_k(\omega) : k \in \mathbb{N})$ such that for $\forall t_1, \dots, t_n \in [0, 1]$,

   $$P(U_1 \leq t_1, \dots, U_n \leq t_n) = \prod_{k=1}^{n} t_k$$

   (a) Show that $(U_k(\omega) : k \in \mathbb{N})$ are independent and uniformly distributed on $[0, 1]$.

   (b) Consider $\overline{U}_n(\omega) = \max\big\{U_1(\omega), \dots, U_n(\omega)\big\}$.
   Compute the cumulative distribution function of $\overline{U}_n$, $F_{\overline{U}_n}(t) = P(\overline{U}_n \leq t)$.

   (c) Show that $\lim_{n\to\infty} \overline{U}_n(\omega) = 1$ $\mathbb{P}$-almost surely.

(d) Let $\underline{U}_n(\omega) = \min\{U_1(\omega), \ldots, U_n(\omega)\}$.
Compute the cumulative distribution function of $\underline{U}_n$, $F_{\underline{U}_n}(t) = P(\underline{U}_n \leq t)$.

(e) Show that $\lim_{n\to\infty} \underline{U}_n(\omega) = 0$ $\mathbb{P}$-almost surely.
Hint: $V_n = (1 - U_n)$ has the same distribution as U_n, which implies that $\underline{U}_n$ and $(1 - \overline{U}_n)$ have the same distribution.

3. (a) let $X(\omega), X_n(\omega), n \in \mathbb{N}$ such that $X_n(\omega) \to X(\omega)$ P-almost surely.
Show that also the Cesaro mean converges P-almost surely to X

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} X_i(\omega) = X(\omega) \qquad P\text{-almost surely}$$

(b) Assume now that $E_P\big(|X_n - X|\big) \to 0$, as $n \to \infty$ (without assuming P-almost sure convergence).
Show that the Cesaro mean is converging in $L^1(P)$, that is

$$\lim_{n\to\infty} E_P\left(\left|\left\{ \frac{1}{n} \sum_{i=1}^{n} X_i \right\} - X(\omega)\right|\right) \to 0 \qquad \text{as } n \to \infty$$

Hint: note that by the triangle inequality

$$\left|\left\{ \frac{1}{n} \sum_{i=1}^{n} X_i \right\} - X(\omega)\right| \leq \frac{1}{n} \sum_{i=1}^{n} \big|X_i - X(\omega)\big| =$$
$$\frac{1}{n} \sum_{i=1}^{M} \big|X_i - X(\omega)\big| + \frac{1}{n} \sum_{j=M+1}^{n} \big|X_j - X(\omega)\big|$$

$\forall n \geq M$, where the inequalities are preserved after taking the expectation.

4. Let $X(\omega), (X_n(\omega) : n \in \mathbb{N})$, random variables on a probability space $(\Omega, \mathcal{F}, P)$.

Show that if $\forall \varepsilon > 0$

$$\sum_{n=0}^{\infty} P(|X_n(\omega) - X(\omega)| > \varepsilon) < \infty$$

it follows $\lim_{n\uparrow\infty} X_n(\omega) = X(\omega)$ P-almost surely.

Hint: show first that

$$\big\{\omega : X_n(\omega) \not\to X(\omega)\big\} = \bigcup_{k\in\mathbb{N}} \{\omega : |X_n(\omega) - X(\omega)| > k^{-1} \text{ infinitely often } \}$$

and recall Borel-Cantelli's lemma.

5. Consider a random variable $X(\omega)$ with $E_P(|X|) < \infty$. Show that

$$E_P\big(|X|\mathbf{1}(|X| > n)\big) = \int_\Omega |X(\omega)|\mathbf{1}\big(|X(\omega)| > n\big) P(d\omega) \to 0 \text{ as } n \to \infty \ .$$