HU, Probability Theory Fall 2015, Problems 6 (14.10.2015)

1. When the cumulative distribution function $F_X(t) = P(X \le t)$ of a $\mathbb{R}$-valued random variable X $F_X(t) = P(X \le t)$ of a $\mathbb{R}$-valued random variable X which is absolutely continuous with respect to Lebesgue measure, which means

$$F_X(b) = F_X(a) + \int_a^b f_X(t)dt$$

for some Borel measurable function $f_X(t) \ge 0$, which is called probability density function. When the classical derivative $\frac{dF_X}{dt}(t)$ exists at all t, then it is a probability density function. More in general $\frac{dF_X}{dt}(t) = \frac{dP_X}{dt}(t) =$ is understood as the Radon Nikodym derivative of the push-forward probability measure P_X with respect to Lebesgue measure.

In such case, for every non-negative and Borel measurable test function $g(x) \ge 0$ we have

$$\begin{aligned} E_P\big(g(X)\big) &= \int_\Omega g(X(\omega))P(d\omega) = \int_{\mathbb{R}} g(t)P_X(dt) \\ &= \int_{\mathbb{R}} g(t)F(dt) = \int_{\mathbb{R}} g(t)f_X(t)dt \end{aligned} \tag{0.1}$$

where $P_X(B) = \mathbb{P}\big(\{\omega : X(\omega) \in B\}\big)$ is the pushforward measure of $\mathbb{P}$ by the random variable X. The integral w.r.t. P_X on $\mathbb{R}$ is the same as the Lebesque Stieltjes integral w.r.t dF, meaning that P_X coincides with the measure induced by the cumulative distribution function $F(t)$ on $\mathbb{R}$.

Hint: One possible strategy for this proof is to use the monotone class theorem: Define the class

$$\mathcal{C} = \big\{g : \mathbb{R} \to [0,\infty) \text{ bounded and Borel measurable such that (0.1) holds } \big\}$$

and show that $\mathcal{C}$ is a monotone class (use the linearity of the integral together with the monotone convergence theorem) which contains the indicators $\mathbf{1}_{(a,b]}(t)$ $\forall a \le b \in \mathbb{R}$.

2. **Linearity of the expectation** The expectation of a random variable $X(\omega)$ is defined as

$$E_{\mathbb{P}}\big(X\big) = E_{\mathbb{P}}\big(X^+\big) - E_{\mathbb{P}}\big(X^-\big)$$

where $X^+ = \max\{X, 0\} \geq 0$, $X^- = \max\{-X, 0\} \geq 0$ are non-negative random variables, and we have defined first for non-negative random variables

$$E_{\mathbb{P}}(X) = \sup_{Y \in \mathcal{S}F: 0 \leq Y \leq X} \{E_{\mathbb{P}}(Y)\}$$

In this way the expectation is well defined unless

$E_{\mathbb{P}}(X^+) = E_{\mathbb{P}}(X^-) = +\infty$.

In the lectures we have shown (first for simple random variables and then by the monotone convergence theorem) that when $X(\omega) \geq 0$, $Y(\omega) \geq 0$ P-almost surely (outside a $\mathbb{P}$-null set), and $a, b \geq 0$

$$E_{\mathbb{P}}(aX + bY) = aE_{\mathbb{P}}(X) + bE_{\mathbb{P}}(Y) \tag{0.2}$$

Show that linearity holds for any random variables X, Y and $a, b \in \mathbb{R}$ when the expectations on both left and right sides in (0.2) are finite.

Hint: write $(aX + bY)$ using the representations $X = (X^+ - X^-)$, $Y = (Y^+ - Y^-)$, $a = (a^+ - a^-)$, $b = (b^+ - b^-)$, and integrate the positive parts and negative parts separately.

3. Let $U(\omega)$ be uniformly distributed r.v. with values in $[0, 1]$, such that $\mathbb{P}(\{U \in (a, b]\}) = (b - a)$ for $0 \leq a \leq b \leq 1$.

   (a) Show that the powers $U(\omega)^z$, with $z \in \mathbb{Z}$ (the integers) are random variables.

   (b) Compute the moments $E_{\mathbb{P}}(U^z) \in [0, +\infty]$ for $z \in \mathbb{Z}$.

   (c) Compute the exponential moments $E_{\mathbb{P}}\big(\exp(tU)\big)$ for $t \in \mathbb{R}$.

   (d) Compute the trigonometric moments $E_{\mathbb{P}}\big(\cos(2\pi tU)\big)$ and $E_{\mathbb{P}}\big(\sin(2\pi tU)\big)$ for $t \in \mathbb{R}$.

4. Let $f : [0, T] \to \mathbb{R}^+$ be a non-negative and bounded measurable function.

   We define its upper and lower Riemann-integrals as follows:

$$J^+(f) = \inf\big\{I(g) : g \geq f, g \text{ takes finitely many values and is piecewise continuous }\big\}$$
$$J^-(f) = \sup\big\{I(g) : g \leq f, g \text{ takes finitely many values and is piecewise continuous }\big\}$$

   where the integral $I(g)$ of a piecewise continuous function g taking finitely many values is the usual finite sum.

Note that on the real line, a piecewise continuous simple function taking finitely many values is piecewise constant, with representation

$$g(x) = \sum_{k=1}^{n} a_k \mathbf{1}_{E_i}(x), \text{ with } I(g) = \sum_{k=1}^{n} a_k \text{ length}(E_i)$$

where E_i are intervals. In the construction of Lebesgue integral, the general definition uses Borel sets instead of intervals.

We say that f is Riemann integrable when $J^+(f) = J^-(f)$ which defines the Riemann integral $J(f)$ (it is possible that $J(f) = +\infty$).

(a) Show that when f is Riemann integrable the Riemann integral $J(f)$ coincides with Lebegue integral $I(f)$ defined in the lectures.
Hint We define the Lebesgue integral $I(f)$ of a Borel measurable non-negative function w.r.t. Lebesgue measure as

$$I(f) = \sup\{I(g) : g \leq f, \quad g \text{ is measurable and takes finitely many values } \}$$

(b) Show that a non-negative continuous function f is Riemann integrable on the compact set $[0, T]$.
Hint: a continuous function uniformly continuous on compact sets. Note that you can approximate uniformly on compacts a continuous function by **piecewise continuous** simple functions.

(c) Let $f(x) = \mathbf{1}_{\mathbb{Q}}(x)$ where $\mathbb{Q}$ are the rationals.
Show that f is Borel measurable, but is not Riemann integrable on $[0, T]$.
Hint : Show that on a compact interval $[0, T]$ $J^+(f) = T$ and $J^-(f) = 0$.

(d) Show that for the Lebesgue integral we have

$$I(f) = \int_0^T f(x)dx = 0$$

5. (a) Prove *Chebychev inequality*: for a random variable X with $X(\omega) \geq 0$ P-almost surely,

$$\mathbb{P}(X > t) \leq \frac{E_{\mathbb{P}}(X)}{t} \quad \forall t > 0$$

Hint Note that

$$0 \leq t\, \mathbf{1}(X(\omega) > t) \leq X(\omega) \,.$$

(b) Prove *Chentsov inequality*

$$\mathbb{P}(X > t) \leq \inf_{\theta > 0} \Big\{ \exp(-\theta t) E_{\mathbb{P}}\big(\exp(\theta X)\big) \Big\}$$

Hint: for any $\theta > 0$, $X > t \Longleftrightarrow \exp(\theta X) > \exp(\theta t)$.

(c) Consider a random variable $N(\omega)$ with Poisson(λ) distribution, where $\lambda > 0$ is the parameter and

$$\mathbb{P}_\lambda\big(N = k\big) = \exp(-\lambda)\frac{\lambda^k}{k!} \quad k \in \mathbb{N} = \{0, 1, 2, \dots\}$$

(d) Knowing that $E(\exp(\theta N)) = \exp\big(\lambda(e^\theta - 1)\big)$, (computed in the exercise sheet n.5) use Chentsov inequality to bound from above the probability $\mathbb{P}_\lambda\big(N > t\big)$, for $t > 0$.