HU, Probability Theory Fall 2015, Problems 5 (7.10.2015)

1. Consider the probability space $\Omega = [0,1]$ equipped with the Borel σ-algebra $\mathcal{F} = \mathcal{B}([0,1])$ and the uniform probability measure $\mathbb{P}$ such that $\mathbb{P}((a,b]) = b - a$ for $0 \le a \le b \le 1$, which is also called Lebesgue measure.

   Show that the identity map $U : \Omega \to [0,1]$ with $U(\omega) = \omega$ is an uniformly distributed random variable, which means

   $\mathbb{P}(\{\ \omega : U(\omega) \in (a,b]\}) = b - a.$

   Let now $(\Omega, \mathcal{F}, \mathbb{P})$ be an abstract probability space and $U : \Omega \to [0,1]$ a random variable with uniform distribution on $[0,1]$, which means $\mathbb{P}(\{\ \omega : U(\omega) \in (a,b]\}) = b - a.$

   Let $F : \mathbb{R} \to [0,1]$ a cumulative probability distribution function (c.d.f.), which is right continuous, non-decreasing with $F(+\infty) = 1$ and $F(-\infty) = 0$.

   We shall construct a random variable on $(\Omega, \mathcal{F})$ with values in $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ such that

$$\mathbb{P}(\{\omega :\ X(\omega) \le t\}) = F(t)$$

   Assume for simplicity that $F(t)$ is continuous and stricty increasing, with $F(s) < F(t)\ \forall s < t$.

   In this case there is an unique inverse $F^{-1} : [0,1] \to \mathbb{R}$ such that $F(F^{-1}(u)) = u\ \forall u \in [0,1]$ and $F^{-1}(F(t)) = t\ \forall t \in \mathbb{R}$.

   Show that $X(\omega) = F^{-1}(U(\omega))$ is a random variable with

   $P(\{X(\omega) \le t\}) = F(t).$

   Using a generalized inverse, this construction extends also to the general cumulative distribution function, which does not need to be continous from the left neither strictly increasing.

2. On an abstract probability space $(\Omega, \mathcal{F}, \mathbb{P})$, let $X(\omega) \ge 0\ \forall \omega \in \Omega$ a non-negative random variable.

   We have defined the expectation of as

$$E_P(X) = \sup_{0 \le Y \le X, \text{ with } Y \in \mathcal{S}F^+} E_P(Y)$$

   where the supremum is taken over the simple random variables Y (taking finitely many values) such that $0 \le Y(\omega) \le X(\omega)\ \forall \omega \in \Omega$

   Assume that $X(\omega) \in \mathbb{N}\ \forall \omega \in \Omega$.

(a) Show that

$$E_P(X) = \sum_{n=1}^{\infty} n\mathbb{P}(\{\ \omega : X(\omega) = n\}) \ = \sum_{n=1}^{\infty} nP_X(\{n\})$$

where $P_X(\{n\}) = \mathbb{P}(\{\omega : X(\omega) = n\})$ is the distribution of X with $E_P(X) \in [0, +\infty]$ (the series may also diverge).

(b) Show one non-trivial example with $X(\omega)$ taking countably many values in $\mathbb{N}$ and choosing the distribution of X $P_X(\{n\})$ such that $E_P(X) < \infty$, and another example where $E_{\mathbb{P}}(X) = +\infty$.

3. On an abstract probability space $(\Omega, \mathcal{F}, \mathbb{P})$,

   let $N(\omega)$ be a Poisson distributed random variable with parameter $\lambda > 0$, such that

$$\mathbb{P}(\{\omega : N(\omega) = k\}) = P_\lambda(\{k\}) = \exp(-\lambda)\frac{\lambda^k}{k!}$$

(a) Check that $(P_\lambda(\{k\}) : k \in \mathbb{N})$ defines a probability distribution on $\mathbb{N} = \{0, 1, 2, \dots\}$, in particular that $P_\lambda(\mathbb{N}) = 1$.

(b) Compute the *moment generating function* $m : \mathbb{R} \to [0, \infty]$

$$m(\theta) = E_{\mathbb{P}}\big(\exp(\theta N)\big), \quad \theta \in \mathbb{R}.$$

(c) Prove the following *Stein equation* for the Poisson distribution:

$$E_P(\lambda g(N + 1)) = E_P(Ng(N))$$

for every bounded sequence $(g_k : k \in \mathbb{N}) \subseteq \mathbb{R}$.

(d) Compute the expectations (moments) $\mathbb{E}_{\mathbb{P}}(N^q)$ for $q \in \mathbb{N}$

(e) Compute the expectations $\mathbb{E}_{\mathbb{P}}\big(N^q \exp(\theta N)\big)$ for $\theta \in \mathbb{R}$ and $q \in \mathbb{N}$.